## 29th Internet Seminar

# Eventual Positivity

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Lecture Notes
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# Contents

# Introduction

## Prerequisites

This course was designed for postgraduate students (Masters and PhD) and advanced Bachelor students with the following prerequisite knowledge:

- Calculus/analysis in one and several variables;
- Linear algebra (in particular eigenvalues and the Jordan normal form of matrices);
- An introduction to real analysis, measure and integration theory (in particular, familiarity with $L^p$ spaces);
- An introduction to functional analysis (Banach spaces, Hilbert spaces, bounded linear operators); and
- An introduction to complex analysis (holomorphic functions, complex path integrals and Cauchy's integral formula, Laurent series).

## Acknowledgements

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# Nomenclature

The following table gives an overview of important symbols used in the lectures. This table will be updated every week as we introduce new notation.

**Elementary notation**

| | |
|---|---|
| $\mathbb{N}$ | strictly positive integers, $\mathbb{N} = \{1,2,3,\dots\}$ |
| $\mathbb{N}_0$ | non-negative integers, $\mathbb{N}_0 = \{0,1,2,\dots\}$ |
| $\mathbb{R}_+$ | alternative notation for the interval $[0,\infty)$ |

**Function spaces**

| | |
|---|---|
| $\mathbb{1}$ | The vector in $\mathbb{R}^n$ whose entries are all 1, or the constant function with value 1 on a set that is clear from the context |
| id | The identity matrix in $\mathbb{C}^{n\times n}$ or the identity operator on a normed space that is clear from the context |

**Spectral theory**

| | |
|---|---|
| $\sigma(A)$ | spectrum of linear operator $A$ |
| $\rho(A)$ | resolvent set of a linear operator $A$, i.e. $\rho(A) := \mathbb{C} \setminus \sigma(A)$ |
| $\mathcal{R}(\lambda, A)$ | resolvent of $A$ at a point $\lambda \in \rho(A)$, i.e. $\mathcal{R}(\lambda, A) := (\lambda - A)^{-1}$ |
| $\mathrm{r}(A)$ | spectral radius of a bounded linear operator $A$, defined by the formula $\mathrm{r}(A) := \max\{\lvert\lambda\rvert : \lambda \in \sigma(A)\} \in [0,\infty)$ |
| $\mathrm{s}(A)$ | spectral bound of a linear operator $A$, defined by the formula $\mathrm{s}(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} \in [-\infty,\infty]$ |

**Ordered structures on vector spaces**

| | |
|---|---|
| $x \preceq y$ | $cx \le y$ for some number $c > 0$ (equivalently, $x \le cy$ for some number $c > 0$) |
| $y \succeq x$ | alternative notation for $x \preceq y$ |

# Chapter 1

# Positive matrices and matrix semigroups

The topic of the ISEM 29 is the interplay between dynamical systems (more specifically: differential equations), sign preservation, and operator theory. The material in the first two chapters develops the essence of the theory in finite dimensions. In Chapter 1, we study positive matrices and matrix exponential functions, and show how the positivity affects their eigenvalues and eigenvectors. The titular subject, *eventual positivity*, makes an appearance in Chapter 2.

## 1.1 Positive matrices and the standard order on $\mathbb{R}^n$

As a foundation for everything that follows, we endow the spaces $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ with the following partial order.

**Definition 1.1.1** (The order and the cone on $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$)**.**

(a) For $x, y \in \mathbb{R}^n$ we write $x \le y$ if this inequality holds componentwise, i.e. if $x_k \le y_k$ for every index $k$. As usual we use the notation $y \ge x$ synonymously with $x \le y$.

Vectors $x \in \mathbb{R}^n$ that satisfy $x \ge 0$ [1] are called the **positive** elements of $\mathbb{R}^n$, and the set $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \ge 0\}$ of all positive vectors is called the **positive cone** of $\mathbb{R}^n$.

(b) We use the same conventions for matrices: for $A, B \in \mathbb{R}^{m\times n}$ we write $A \le B$ (or $B \ge A$) if $A_{jk} \le B_{jk}$ for all indices $j, k$.

A matrix $A \in \mathbb{R}^{m\times n}$ is called **positive** if $A \ge 0$, and the set $\mathbb{R}^{m\times n}_+ := \{A \in \mathbb{R}^{m\times n} : A \ge 0\}$ of positive matrices is called the **positive cone** in $\mathbb{R}^{m\times n}$.

Note that Definition 1.1.1(a) can be considered a special case of part (b) if we identify $\mathbb{R}^n$ with $\mathbb{R}^{n\times 1}$. The relation $\le$ is a partial order on $\mathbb{R}^n$ and is compatible with its vector

[1] We always write 0 for the zero vector when the corresponding space is clear from context.

space structure in the following sense: if $x \le y$ for $x, y \in \mathbb{R}^n$, then

$$\alpha x \le \alpha y \qquad \text{and} \qquad x + z \le y + z$$

for all numbers $\alpha \in [0, \infty)$ and all vectors $z \in \mathbb{R}^n$. Analogous statements hold for the partial order $\le$ on $\mathbb{R}^{m\times n}$.

**Remark 1.1.2** (Terminology: positive vectors)**.** At first glance, it might be surprising that our definition of 'positivity' is inconsistent with its common meaning for real numbers: in English, a number $\alpha \in \mathbb{R}$ is usually called positive if $\alpha > 0$. Yet, the vector $0 \in \mathbb{R}^n$ is positive in the sense of Definition 1.1.1. For $n = 1$ this means, in particular, that the real number 0 is positive in the sense of Definition 1.1.1.

Nevertheless, our usage of 'positive' is standard in the theory of Banach lattices, which we use frequently from Chapter 4 on. For readers who take pleasure in terminological digressions, a more thorough discussion is provided in the notes at the end of this chapter.

From an operator-theoretic perspective, it is desirable to describe positivity of matrices in terms of how they act as linear maps. We do this in the next proposition.

**Proposition 1.1.3.** *For a matrix $A \in \mathbb{R}^{m\times n}$, the following are equivalent:*

(i) *$A$ is positive, i.e. $A \in \mathbb{R}_+^{m\times n}$.*

(ii) *$A(\mathbb{R}_+^n) \subseteq \mathbb{R}_+^m$.*

(iii) *$A$ acts monotonically, i.e. if $x, y \in \mathbb{R}^n$ satisfy $x \le y$, then $Ax \le Ay$.*

*Proof.* *"(i) ⇒ (ii)"*: This is clear from the definition of the matrix-vector product.

*"(ii) ⇒ (iii)"*: Assume that (ii) holds and let $x, y \in \mathbb{R}^n$ satisfy $x \le y$. Then $y - x \in \mathbb{R}_+^n$ and hence $Ay - Ax = A(y - x) \in \mathbb{R}_+^m$, which implies that $Ax \le Ay$.

*"(iii) ⇒ (i)"*: Assume that (iii) holds. For $j \in \{1, \dots, n\}$ and the canonical unit vector $e_j \in \mathbb{R}^n$ one has $0 \le e_j$ and thus $0 = A0 \le Ae_j$. Since $Ae_j$ is the $j$-th column of $A$ and $j$ was arbitrary, we conclude that all entries of $A$ are $\ge 0$. □

Since we defined the order relation $\le$ by comparing vectors (and matrices) entrywise, it is natural to generalise the modulus from scalars to vectors in the same way:

**Definition 1.1.4** (The modulus of vectors and matrices)**.** For every vector $x \in \mathbb{C}^n$ and every matrix $A \in \mathbb{C}^{m\times n}$ we define the matrix $|A| \in \mathbb{R}_+^{m\times n}$ and the vector $|x| \in \mathbb{R}_+^n$ by taking the entrywise modulus of $x$ and $A$, i.e.

$$|x|_j := \left|x_j\right| \qquad \text{and} \qquad |A|_{jk} := \left|A_{jk}\right|$$

for all indices $j$ and $k$.

The modulus has a submultiplicative property, which is very useful to prove estimates for positive matrices.

**Proposition 1.1.5** (Submultiplicativity of the modulus)**.** *Let $A \in \mathbb{C}^{m\times n}$ and $x \in \mathbb{C}^n$.*

(a) *One has $|Ax| \le |A|\,|x|$.*

(b) *In particular, if $A \in \mathbb{R}_+^{m\times n}$, then $|Ax| \le A|x|$.*

*Proof.* (a) One can check the inequality entrywise: for every $j \in \{1,\dots,n\}$ one has

$$|Ax|_j = \left|(Ax)_j\right| = \left|\sum_{k=1}^n A_{jk}x_k\right| \le \sum_{k=1}^n \left|A_{jk}\right| |x_k| = \big(|A|\,|x|\big)_j \,.$$

(b) For positive $A$, one has $|A| = A$, so the claim follows from part (a). ☐

**Remark 1.1.6** (Norms on $\mathbb{C}^n$)**.** In the following we often work with norms on $\mathbb{C}^n$. While they are all equivalent, we assume throughout that $\mathbb{C}^n$ is endowed with a norm that satisfies $\||x|\| = \|x\|$ for all $x \in \mathbb{C}^n$ as well as $\|x\| \le \|y\|$ for all $x, y \in \mathbb{R}^n$ with $0 \le x \le y$ – this is sometimes more convenient in estimates. For instance, the $p$-norm has this property for every $p \in [1,\infty]$.

## 1.2 The spectrum of positive matrices

An intriguing feature of positive matrices is that their eigenvalues and eigenvectors enjoy a variety of remarkable properties. This is the content of the classical **Perron-Frobenius theorem**, which we study in this section. This theorem is only a first instance of one of the most important themes of the course: the interaction between positivity and the spectrum of linear operators. We take this opportunity to introduce some fundamental concepts and tools in spectral theory.

**Definition 1.2.1** (Spectrum and spectral radius)**.** Let $A \in \mathbb{C}^{n\times n}$. The set $\sigma(A) \subseteq \mathbb{C}$ that consists of all eigenvalues of $A$ is called the **spectrum** of $A$, and the number

$$\mathrm{r}(A) := \max\big\{|\lambda| : \lambda \in \sigma(A)\big\} \in [0,\infty)$$

is called the **spectral radius** of $A$.

The spectral radius determines whether the powers of a matrix converge to 0 as the exponent tends to $\infty$. More precisely, one has the following equivalence.

**Proposition 1.2.2** (Convergence to 0 of matrix powers)**.** *For every matrix $A \in \mathbb{C}^{n\times n}$, the following are equivalent:*

(i) $\mathrm{r}(A) < 1$.

(ii) *$A^k \to 0$ as $k \to \infty$.*

(iii) *There exist numbers $\eta \in [0,1)$ and $c \ge 0$ such that $\left\|A^k\right\| \le c\eta^k$ for each $k \in \mathbb{N}_0$.*

*Proof.* "(i) ⇒ (iii)": The implication is clear if $\mathrm{r}(A) = 0$, hence we assume $\mathrm{r}(A) > 0$. One can then show, using the Jordan normal form of $A$, that $\|A^k\| \le \tilde{c}\,\mathrm{r}(A)^k(1 + k^{n-1})$ for a number $\tilde{c} \ge 0$ and all $k \in \mathbb{N}_0$; see Exercise 1.4(c). So the claim follows by taking any $\eta \in (\mathrm{r}(A), 1)$ and using that $\frac{\mathrm{r}(A)^k}{\eta^k}$ decays exponentially.

"(iii) ⇒ (ii)": This implication is obvious.

"(ii) ⇒ (i)": Let $\lambda$ be an eigenvalue of $A$ with $|\lambda| = \mathrm{r}(A)$ associated to an eigenvector $z$ of norm one. One has $|\lambda|^k = |\lambda|^k \|z\| = \|A^k z\| \to 0$ as $k \to \infty$. Thus, $\mathrm{r}(A) = |\lambda| < 1$. □

To formulate the statement of some parts of the Perron-Frobenius theorem we need the following stronger notion of positivity.

**Definition 1.2.3** (Strong positivity in finite dimensions)**.** A vector $x \in \mathbb{R}^n$ is called **strongly positive** if $x_k > 0$ for all $k \in \{1, \dots, n\}$. Similarly, a matrix $A \in \mathbb{R}^{m \times n}$ is called **strongly positive** if $A_{jk} > 0$ for all indices $j, k$.

Observe that the strongly positive vectors in $\mathbb{R}^n$ are precisely the points in the interior of the positive cone $\mathbb{R}^n_+$. Similarly as in Proposition 1.1.3, strong positivity of matrices can also be interpreted in terms of their actions as linear mappings: a matrix $A \in \mathbb{R}^{m \times n}$ is strongly positive if and only if it maps every $0 \ne x \in \mathbb{R}^n_+$ to a strongly positive vector.

It is convenient to have a notation for strong positivity. The following has the advantage that it can easily be generalised to the infinite-dimensional setting in later chapters.

**Notation 1.2.4** (Inequality up to a factor)**.**

(a) For two vectors $x, y \in \mathbb{R}^n$ we write $x \preceq y$ or equivalently $y \succeq x$ if there exists a number $c > 0$ such that $cx \le y$ (equivalently, if there exists a number $c > 0$ such that $x \le cy$).

(b) We let $\mathbb{1} \in \mathbb{R}^n$ denote the vector with every entry equal to 1. Hence, a vector $x \in \mathbb{R}^n$ is strongly positive if and only if $x \succeq \mathbb{1}$.

The main result of this section is the following classical theorem about the eigenvalues and eigenvectors of positive matrices.

**Theorem 1.2.5** (Perron–Frobenius)**.** *Let* $0 \le A \in \mathbb{R}^{n \times n}$.

(a) *The spectral radius* $\mathrm{r}(A)$ *is an eigenvalue of* $A$ *with an eigenvector* $x \ge 0$.

(b) *If all diagonal entries of* $A$ *are non-zero, then* $\mathrm{r}(A) > 0$*, and* $\mathrm{r}(A)$ *is a* ***radially strictly dominant*** *eigenvalue in the sense that* $|\lambda| < \mathrm{r}(A)$ *for all other eigenvalues* $\lambda$ *of* $A$.

(c) *If* $A$ *is even strongly positive, then* $\mathrm{r}(A) > 0$*, the eigenvalue* $\mathrm{r}(A)$ *of* $A$ *is algebraically simple,*[2] *and its eigenspace is spanned by a strongly positive vector.*

[2] Recall that the **algebraic multiplicity** of an eigenvalue $\lambda$ of $A$ is the dimension of the generalised eigenspace $\bigcup_{k=1}^n \ker(\lambda - A)^k$. The eigenvalue $\lambda$ is called **algebraically simple** if its algebraic multiplicity is one.

The Perron–Frobenius theorem is quite useful to study the behaviour of the powers $A^k$ of a positive matrix $A$ as $k \to \infty$. We explore a concrete application to Markov chains in Exercise 1.3.

Various proofs of the theorem and variations thereof are known; see e.g. the survey article [Mac00] for some nice bedtime reading. The proof we present has a strong functional analytic flavour and already anticipates several ideas and arguments that occur again in the infinite-dimensional case – strongly relying on properties of the resolvent. We define and study this object now and finally use it to prove Theorem 1.2.5.

**Definition 1.2.6** (The resolvent of a matrix)**.** Let $A \in \mathbb{C}^{n\times n}$. The complement of its spectrum, i.e. $\rho(A) := \mathbb{C} \setminus \sigma(A)$, is called the **resolvent set** of $A$. The mapping

$$\mathcal{R}(\cdot, A)\colon \rho(A) \to \mathbb{C}^{n\times n}, \qquad \lambda \mapsto \mathcal{R}(\lambda, A) := (\lambda - A)^{-1}$$

is called the **resolvent** of $A$.

In the preceding definition, we used the notation $\lambda - A$, which is shorthand for $\lambda\,\mathrm{id} - A$; where $\mathrm{id} \in \mathbb{C}^{n\times n}$, denotes the identity matrix of the same dimension as $A$.

To state the following proposition we need the concept of a vector-valued analytic functions. In finite dimensions this is easy: a mapping from an open subset of $\mathbb{C}$ to $\mathbb{C}^n$ or to $\mathbb{C}^{n\times n}$ is called **analytic** or **holomorphic** if every component of the mapping is analytic.

**Proposition 1.2.7** (Properties of resolvents)**.** *Let $A \in \mathbb{C}^{n\times n}$.*

(a) *The resolvent $\mathcal{R}(\cdot, A)\colon \rho(A) \to \mathbb{C}^{n\times n}$ is analytic.*

(b) *For $\lambda \in \mathbb{C}$ with $|\lambda| > \mathrm{r}(A)$, the resolvent can be represented as the* ***Neumann series***

$$\mathcal{R}(\lambda, A) = \sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}},$$

*which converges absolutely in $\mathbb{C}^{n\times n}$ (with respect to any norm).*

*Proof.* (a) It follows from Cramer's rule for the inverse of a matrix that, for all indices $j,k$, the matrix entry $\mathcal{R}(\cdot, A)_{jk}\colon \rho(A) \to \mathbb{C}$ is a rational function and thus analytic.

(b) As $\mathrm{r}(A/\lambda) < 1$, so by Proposition 1.2.2, there exist numbers $\eta \in [0,1)$ and $c \geq 0$ such that $\|A^k/\lambda^k\| \leq c\eta^k$ for every $k \in \mathbb{N}_0$. Thus, $\sum_{k=0}^{\infty} \left\| \frac{A^k}{\lambda^{k+1}} \right\| < \infty$, and hence the series converges absolutely in $\mathbb{C}^{n\times n}$. To show the resolvent formula, we compute

$$(\lambda - A)\sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}} = \lim_{K\to\infty} \sum_{k=0}^{K} \left( \frac{A^k}{\lambda^k} - \frac{A^{k+1}}{\lambda^{k+1}} \right) = \lim_{K\to\infty} \left( \mathrm{id} - \frac{A^{K+1}}{\lambda^{K+1}} \right) = \mathrm{id}.$$

Here we used that $\|A^{k+1}/\lambda^{k+1}\| \to 0$ as $k \to \infty$ according to Proposition 1.2.2 since $\mathrm{r}(A/\lambda) < 1$. After multiplying by $\mathcal{R}(\lambda, A)$, we obtain the claimed formula. □

Finally, we need the following lemma about simplicity of eigenvalues. It is illuminating to check explicitly how the assumption $v^{\mathrm{T}}u \neq 0$ below fails for a $2 \times 2$ Jordan block.

**Lemma 1.2.8** (Algebraic simplicity from geometric simplicity)**.** *Let $\lambda \in \mathbb{C}$ be a geometrically simple eigenvalue*[3] *of some $A \in \mathbb{C}^{n\times n}$. If there exist eigenvectors $u$ and $v$ of $A$ and $A^{\mathrm{T}}$ respectively for the eigenvalue $\lambda$ satisfying $v^{\mathrm{T}} u \neq 0$, then $\lambda$ is even an algebraically simple eigenvalue of $A$.*

*Proof.* Let $x \in \mathbb{C}^n$. It suffices to show that if $(\lambda - A)^2 x = 0$, then $(\lambda - A)x = 0$, so assume that $(\lambda - A)^2 x = 0$. Since $(\lambda - A)x$ is in the eigenspace $\ker(\lambda - A)$ which is spanned by $u$, there exists a scalar $\alpha \in \mathbb{C}$ such that $(\lambda - A)x = \alpha u$. Hence,

$$\alpha v^{\mathrm{T}} u = v^{\mathrm{T}}(\lambda - A)x = \left(\left(\lambda - A^{\mathrm{T}}\right) v\right)^{\mathrm{T}} x = 0.$$

As $v^{\mathrm{T}} u \neq 0$, this implies that $\alpha = 0$, so $(\lambda - A)x = 0$, as claimed. □

Now we have all the tools that we need to prove the Perron–Frobenius theorem.

*Proof of Theorem 1.2.5.* (a) We first consider the case $\sigma(A) = \{0\}$. In this case, one has $\mathrm{r}(A) = 0 \in \sigma(A)$. Moreover, there exists an integer $k \geq 1$ such that $A^k = 0$. Choose any non-zero vector $y \in \mathbb{R}^n_+$ and let $j \in \{0, 1, \dots, k-1\}$ be the maximal number for which $x := A^j y \neq 0$. Then $x$ is positive since $A^j$ is positive, and $x \in \ker A$.

Now we consider the more interesting case where $\sigma(A) \neq \{0\}$ and hence $\mathrm{r}(A) > 0$.

Choose an eigenvalue $\lambda$ of $A$ with modulus $|\lambda| = \mathrm{r}(A)$ and let $z \in \mathbb{C}^n$ be an eigenvector of norm 1 corresponding to $\lambda$. For every $s > 1$ one has $\mathcal{R}(s\lambda, A)z = \frac{1}{s\lambda - \lambda} z$, and thus

$$\begin{aligned}\frac{1}{(s-1)\,\mathrm{r}(A)}\,|z| = \left|\frac{1}{s\lambda - \lambda} z\right| = |\mathcal{R}(s\lambda, A)z| &= \left|\sum_{k=0}^{\infty} \frac{A^k}{(s\lambda)^{k+1}} z\right| \\ &\leq \sum_{k=0}^{\infty} \frac{\left|A^k\right|}{|s\lambda|^{k+1}}\,|z| = \sum_{k=0}^{\infty} \frac{A^k}{(s\,\mathrm{r}(A))^{k+1}}\,|z| = \mathcal{R}(s\,\mathrm{r}(A), A)\,|z|;\end{aligned}$$

where the penultimate equality uses the positivity of $A^k$ (Proposition 1.1.5). Here we have twice used the Neumann series representation of the resolvent (Proposition 1.2.7(b)), which is applicable because $|s\lambda|, |s\,\mathrm{r}(A)| > \mathrm{r}(A)$.

If we take norms in the inequality $\frac{1}{(s-1)\,\mathrm{r}(A)}\,|z| \leq \mathcal{R}(s\,\mathrm{r}(A), A)\,|z|$ that we just proved, we get $\frac{1}{(s-1)\,\mathrm{r}(A)} \leq \|\mathcal{R}(s\,\mathrm{r}(A), A)\|$ (see the properties of the norm in Remark 1.1.6), so $\|\mathcal{R}(s\,\mathrm{r}(A), A)\| \to \infty$ as $s \downarrow 1$. By continuity of the resolvent (Proposition 1.2.7(a)), it follows that $\mathrm{r}(A)$ is not in the resolvent set and is thus an eigenvalue of $A$.

It remains to show the existence of an eigenvector $x \in \mathbb{R}^n_+$ for the eigenvalue $\mathrm{r}(A)$. Consider any sequence $(s_k)$ in $(1, \infty)$ that converges to 1; for each index $k$ we define

$$\alpha_k := \|\mathcal{R}(s_k\,\mathrm{r}(A), A)\,|z|\| \qquad \text{and} \qquad x_k := \frac{\mathcal{R}(s_k\,\mathrm{r}(A), A)\,|z|}{\alpha_k}.$$

[3] Recall that the **geometric multiplicity** of an eigenvalue $\lambda$ of $A$ is the dimension of the eigenspace $\ker(\lambda - A)$. The eigenvalue $\lambda$ is called **geometrically simple** if its geometric multiplicity is one.

We have already seen that $\alpha_k \to \infty$ as $k \to \infty$ and that $x_k \geq 0$ for each $k$. Moreover,

$$\begin{aligned}(A - \mathrm{r}(A))x_k &= (A - s_k \mathrm{r}(A))x_k + (s_k \mathrm{r}(A) - \mathrm{r}(A))x_k \\ &= -\frac{|z|}{\alpha_k} + (s_k - 1)\,\mathrm{r}(A)x_k \to 0.\end{aligned}$$

Since $\|x_k\| = 1$ for all $k$ and as the unit sphere in $\mathbb{C}^n$ is compact, there exists a subsequence $(x_{k_j})$ of $(x_k)$ that converges to a non-zero vector $x \in \mathbb{R}^n_+$. Thus, $(A - \mathrm{r}(A))x = \lim_{j\to\infty}(A - \mathrm{r}(A))x_{k_j} = 0$, and so $x$ is an eigenvector of $A$ for the eigenvalue $\mathrm{r}(A)$.

(b) Assume now that all diagonal entries of $A$ are non-zero. Then we can find a number $\delta > 0$ such that $A - \delta \geq 0$.[4] Consider the spectral radius $r := \mathrm{r}(A - \delta)$ of $A - \delta$. Since $A - \delta$ is positive, we can apply (a) to this matrix and thus see that $r$ is an eigenvalue of $A - \delta$ and so $r + \delta$ is an eigenvalue of $A$. In particular, $0 < r + \delta \leq \mathrm{r}(A)$.

On the other hand, as all eigenvalues of $A - \delta$ are contained in the closed disk $B_{\leq r}(0)$ with radius $r$ and centre 0, so all eigenvalues of $A$ are contained in the disk $B_{\leq r}(\delta)$ with radius $r$ and centre $\delta$. Therefore, $\mathrm{r}(A) \leq r + \delta$. It follows that $\mathrm{r}(A) = r + \delta$. But the circle with radius $r + \delta$ and centre 0 intersects the disk $B_{\leq r}(\delta)$ only in the point $r + \delta$, so $A$ has no further eigenvalue of modulus $r + \delta = \mathrm{r}(A)$.

(c) Finally, assume that $A$ is strongly positive. For every eigenvector $x \in \mathbb{R}^n_+$ of $A$ corresponding to the eigenvalue $\mathrm{r}(A)$ – which exists according to (a) – one has $\mathrm{r}(A)x = Ax \succeq \mathbb{1}$. Hence $\mathrm{r}(A) > 0$ and $x \succeq \mathbb{1}$.

Next we show that the eigenvalue $\mathrm{r}(A)$ is geometrically simple. To this end, let $x \succeq \mathbb{1}$ be an eigenvector of $A$ for the eigenvalue $\mathrm{r}(A)$ and let $y \in \mathbb{R}^n$ be any other eigenvector for the same eigenvalue. Then there exists a number $\gamma \in \mathbb{R} \setminus \{0\}$ such that $x - \gamma y$ is positive, but has at least one component that is 0. If $x - \gamma y$ were non-zero, it would be an eigenvector of $A$ for the eigenvalue $\mathrm{r}(A)$, which would imply $x - \gamma y \succeq \mathbb{1}$, as we have just seen. Thus, $x - \gamma y = 0$, so $y$ is a multiple of $x$. This proves the geometric simplicity of the eigenvalue $\mathrm{r}(A)$.

To see that $\mathrm{r}(A)$ is algebraically simple, we now use Lemma 1.2.8. By applying (a) to the transposed matrix $A^{\mathrm{T}}$, one gets an eigenvector $y \geq 0$ of $A^{\mathrm{T}}$ for the eigenvalue $\mathrm{r}(A^{\mathrm{T}}) = \mathrm{r}(A)$. As $y \neq 0$ and $x \succeq \mathbb{1}$, one has $y^{\mathrm{T}}x > 0$, so Lemma 1.2.8 is applicable and shows that the geometric simplicity of $\mathrm{r}(A)$ implies the algebraic simplicity. □

## 1.3 Positive matrix semigroups

The powers $A^k$ of a square matrix give the solutions $x\colon \mathbb{N}_0 \to \mathbb{C}^n$ to the difference equation $x(k) = Ax(k-1)$ for $k \in \mathbb{N}$. As in the scalar case, it is natural to study the continuous time analogue of this dynamical system, i.e. the differential equation $\dot{x}(t) = Ax(t)$ with $x\colon [0,\infty) \to \mathbb{C}^n$. For this, one uses the matrix exponential function.

[4] Let us recall here the convention $A - \delta := A - \delta\,\mathrm{id}$ that we first used Definition 1.2.6.

**Definition 1.3.1** (Matrix exponential function)**.** For every $A \in \mathbb{C}^{n\times n}$ one defines

$$e^A := \exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!} \in \mathbb{C}^{n\times n},$$

where the series converges absolutely in $\mathbb{C}^{n\times n}$.

We first discuss a number of essential properties of the matrix exponential function, in particular its relation to linear differential equations. Positivity takes the stage back in Theorems 1.3.8 and 1.3.9.

**Proposition 1.3.2** (Properties of the matrix exponential function)**.** *The matrix exponential function has the following properties:*

(a) $e^0 = \mathrm{id}$.

(b) *The matrix exponential function* $\exp\colon \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$, $A \mapsto e^A$ *is continuous.*

(c) *For fixed* $A \in \mathbb{C}^{n\times n}$, *the mapping* $\mathbb{C} \to \mathbb{C}^{n\times n}$, $z \mapsto e^{zA}$ *is differentiable, and hence analytic, with derivative* $\frac{\mathrm{d}}{\mathrm{d}z} e^{zA} = Ae^{zA} = e^{zA}A$ *at each* $z \in \mathbb{C}$.

(d) *If two matrices* $A, B \in \mathbb{C}^{n\times n}$ *satisfy* $AB = BA$, *then* $e^{A+B} = e^A e^B$.

*Proof.* (a) This follows readily from the definition of the matrix exponential function.

(b) Let $A, B \in \mathbb{C}^{n\times n}$. An induction argument yields the geometric sum formula

$$A^k - B^k = \sum_{j=0}^{k-1} A^j (A-B) B^{k-1-j}$$

for all integers $k \geq 1$. On the right hand side, it is important to have $A - B$ in the middle since $A$ and $B$ are not assumed to commute. Thus we can estimate $\left\|A^k - B^k\right\| \leq k\alpha^{k-1} \|A - B\|$ with $\alpha := \max\{\|A\|, \|B\|\}$. The continuity now follows from

$$\left\|e^A - e^B\right\| \leq \sum_{k=1}^{\infty} \frac{1}{k!} \left\|A^k - B^k\right\| \leq \|A - B\| \sum_{k=1}^{\infty} \frac{\alpha^{k-1}}{(k-1)!} = e^{\alpha} \|A - B\|.$$

(c) This can be shown in the same way as for the scalar-valued exponential function.

(d) One can prove this by using the Cauchy product formula for infinite series, as in the scalar-valued case. Readers familiar with the uniqueness theorem for ordinary differential equations might also find the following alternative proof insightful.

Consider the functions $X_1, X_2\colon \mathbb{R} \to \mathbb{C}^{n\times n}$ that are given by

$$X_1(t) = e^{t(A+B)} \qquad \text{and} \qquad X_2(t) = e^{tA}e^{tB}$$

for all $t \in \mathbb{R}$. According to (a) and (c) the function $X_1$ solves the initial value problem

$$\begin{cases} \dot{X}(t) = (A+B)X(t) & \text{for all } t \in \mathbb{R}, \\ X(0) = \mathrm{id}. \end{cases}$$

On the other hand, as $A$ and $B$ commute, the definition of the matrix exponential function implies that $B$ also commutes with $e^{tA}$ for all $t \in \mathbb{R}$. This together with (c) and the product rule for differentiation implies that $X_2$ solves the same initial value problem. Hence, by the uniqueness theorem for linear initial value problems it follows that $X_1(t) = X_2(t)$ for all $t \in \mathbb{R}$. For $t = 1$ this gives the claim. □

Proposition 1.3.2(c) has the following consequence, which is the main reason why one is interested in matrix exponential functions.

**Corollary 1.3.3** (Solutions to linear differential equations). *Let $A \in \mathbb{C}^{n\times n}$ and $x_0 \in \mathbb{C}^n$. Then the function $x : [0,\infty) \to \mathbb{C}^{n\times n}$, $t \mapsto e^{tA}x_0$ satisfies the initial value problem*

$$\begin{cases} \dot{x}(t) = Ax(t) & \textit{for all } t \in [0,\infty), \\ x(0) = x_0. \end{cases}$$

From the uniqueness theorem for ordinary differential equations, the function $x$ is in fact the only solution to the initial value problem in Corollary 1.3.3.

For a matrix $A \in \mathbb{C}^{n\times n}$, Corollary 1.3.3 shows that the matrix family $(e^{tA})_{t\geq 0}$ is a quite fundamental object. Hence, it gets its own name, which is inspired by the property $e^{(s+t)A} = e^{sA}e^{tA}$ for all $s, t \geq 0$ that follows from Proposition 1.3.2(d).

**Definition 1.3.4** (Matrix semigroups and positivity).

(a) Let $A \in \mathbb{C}^{n\times n}$. The family $(e^{tA})_{t\geq 0}$ is called the **matrix semigroup** generated by $A$.

(b) Let $A \in \mathbb{R}^{n\times n}$. Then $(e^{tA})_{t\geq 0}$ is called **positive** if $e^{tA} \geq 0$ for all $t \in [0,\infty)$.

We have seen (in Proposition 1.2.2) that the spectral radius of a matrix $A$ determines the long-term behaviour of the powers $A^k$. For the matrix semigroup $(e^{tA})_{t\geq 0}$, a similar role is played by the so-called **spectral bound**.

**Definition 1.3.5** (The spectral bound of a matrix). Let $A \in \mathbb{C}^{n\times n}$. The number

$$\mathrm{s}(A) := \max\left\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\right\}$$

is called the **spectral bound** of $A$.

**Proposition 1.3.6** (Convergence to 0 of matrix semigroups). *For each matrix $A \in \mathbb{C}^{n\times n}$, the following are equivalent:*

(i) $\mathrm{s}(A) < 0$.

(ii) *$e^{tA} \to 0$ as $t \to \infty$.*

(iii) *There exist numbers* $\mu < 0$ *and* $c \geq 0$ *such that* $\|e^{tA}\| \leq ce^{t\mu}$ *for each* $t \geq 0$.

*Proof.* "(i) ⇒ (iii)": As in the proof of Proposition 1.2.2, this can be deduced using the Jordan normal form of $A$. We refer to Exercise 1.4(d) for a detailed discussion; cf. proof of Proposition 1.2.2.

"(iii) ⇒ (ii)": This implication is obvious.

"(ii) ⇒ (i)": Let $\lambda$ be an eigenvalue of $A$ with real part $\operatorname{Re}\lambda = \mathrm{s}(A)$ and an associated eigenvector $z \in \mathbb{C}^n$ of norm one. For every $k \in \mathbb{N}_0$ one has $A^k z = \lambda^k z$ and thus, $e^{tA}z = e^{t\lambda}z$ for every $t \geq 0$ by the definition of the matrix exponential function. So $e^{t\mathrm{s}(A)} = e^{t\operatorname{Re}\lambda}\|z\| = \|e^{tA}z\| \to 0$ for each $t \to \infty$, which shows that $\mathrm{s}(A) < 0$. □

The Neumann series representation of the resolvent of a matrix $A$ (given in Proposition 1.2.7(b)) has the following analogue in continuous time.

**Lemma 1.3.7** (Laplace transform representation of the resolvent)**.** *Let* $A \in \mathbb{C}^{n\times n}$*. For every* $\lambda \in \mathbb{C}$ *that satisfies* $\operatorname{Re}\lambda > \mathrm{s}(A)$ *one has*

$$\mathcal{R}(\lambda, A) = \int_0^\infty e^{-t\lambda}e^{tA}\,\mathrm{d}t,$$

*where the integral converges absolutely.*

*Proof.* Let $\operatorname{Re}\lambda > \mathrm{s}(A)$. Then $\mathrm{s}(A-\lambda) < 0$ and so by Proposition 1.3.6, there are numbers $\mu < 0$ and $c \geq 0$ such that $\|e^{-t\lambda}e^{tA}\| \leq ce^{t\mu}$ for all $t \geq 0$. Hence, the integral indeed converges absolutely.

To prove that the integral equals $\mathcal{R}(\lambda, A)$, observe that

$$(\lambda - A)\int_0^\infty e^{-t\lambda}e^{tA}\,\mathrm{d}t = \lim_{T\to\infty} -\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}e^{t(A-\lambda)}\,\mathrm{d}t = \lim_{T\to\infty}\big(-e^{T(A-\lambda)} + \mathrm{id}\big) = \mathrm{id};$$

where the last equality uses again that $\mathrm{s}(A-\lambda) < 0$, which indeed gives $e^{T(A-\lambda)} \to 0$ as $T \to \infty$ according to Proposition 1.3.6. □

Except in special cases in small dimensions, it is typically not possible to explicitly compute $e^{tA}$ for a given matrix $A$. Fortunately, one can check positivity of the semigroup $(e^{tA})_{t\geq 0}$ purely in terms of $A$, as condition (iv) in the following theorem shows.

**Theorem 1.3.8** (Characterisation of positive matrix semigroups)**.** *Let* $A \in \mathbb{R}^{n\times n}$*. The following are equivalent:*

(i) $e^{tA} \geq 0$ *for all real numbers* $t \geq 0$.

(ii) *For all real numbers* $\lambda > \mathrm{s}(A)$ *one has* $\mathcal{R}(\lambda, A) \geq 0$.

(iii) *For all sufficiently large real numbers* $\lambda > \mathrm{s}(A)$ *one has* $\mathcal{R}(\lambda, A) \geq 0$.

(iv) *All off-diagonal entries of* $A$ *are in* $[0,\infty)$*, i.e.* $A_{jk} \geq 0$ *for all indices* $j \neq k$.

*Proof.* "(i) $\Rightarrow$ (ii)": This follows from the representation of the resolvent $\mathcal{R}(\lambda, A)$ as the Laplace transform of the semigroup $(e^{tA})_{t\geq 0}$ given in Lemma 1.3.7.

"(ii) $\Rightarrow$ (iii)": This implication is obvious.

"(iii) $\Rightarrow$ (iv)": Consider numbers $\lambda \in \mathbb{R}$ that satisfy $\lambda > \mathrm{r}(A)$. The Neumann series representation of the resolvent (Proposition 1.2.7(b)) shows that

$$\lambda^2 \mathcal{R}(\lambda, A) - \lambda \,\mathrm{id} = \sum_{k=1}^{\infty} \frac{A^k}{\lambda^{k-1}} \to A$$

as $\lambda \to \infty$. For indices $j \neq k$ one thus gets

$$A_{jk} = \lim_{\lambda\to\infty} \big(\lambda^2 \mathcal{R}(\lambda, A) - \lambda \,\mathrm{id}\big)_{jk} = \lim_{\lambda\to\infty} \lambda^2 \mathcal{R}(\lambda, A)_{jk} \geq 0,$$

where the last inequality follows from (iii).

"(iv) $\Rightarrow$ (i)": As (iv) holds there exists a number $c \in \mathbb{R}$ such that $A + c\,\mathrm{id} \geq 0$. So

$$0 \leq e^{t(A+c\,\mathrm{id})} = e^{tc\,\mathrm{id}} e^{tA} = e^{tc} e^{tA}$$

for all $t \in [0,\infty)$, where the inequality at the beginning follows from $A + c\,\mathrm{id} \geq 0$ and the definition of matrix exponential function, and the first equality follows from Proposition 1.3.2(d). Division by the numbers $e^{tc} \in (0,\infty)$ yields (i). $\square$

Several other equivalent conditions for positivity of matrix semigroups can be found in Exercise 1.2. We conclude this lecture with a Perron–Frobenius type theorem for positive matrix semigroups. It is remarkable that in part (b) of the following theorem, no additional assumption on $A$ is needed. This is in sharp contrast to situation for single operators, where we needed an additional assumption in Theorem 1.2.5(b).

**Theorem 1.3.9** (Perron–Frobenius for positive matrix semigroups)**.** *Let $A \in \mathbb{R}^{n\times n}$ and assume the matrix semigroup $(e^{tA})_{t\geq 0}$ is positive.*

(a) $\mathrm{s}(A)$ *is an eigenvalue of $A$ and there exists a corresponding eigenvector $x \geq 0$.*

(b) $\mathrm{s}(A)$ *is a **strictly dominant** eigenvalue of $A$ in the sense that* $\operatorname{Re}\lambda < \mathrm{s}(A)$ *for all* $\lambda \in \sigma(A) \setminus \{\mathrm{s}(A)\}$.

*Proof.* Since $(e^{tA})_{t\geq 0}$ is positive, there exists $c \in \mathbb{R}$ such that $A + c \geq 0$ by Theorem 1.3.8. Therefore by the Perron–Frobenius theorem for positive matrices (Theorem 1.2.5), the spectral radius $\mathrm{r}(A+c)$ is an eigenvalue of $A+c$ with a positive eigenvector. Consequently, it equals $\mathrm{s}(A+c)$ and is a strictly dominant eigenvalue of $A+c$. The assertions thus follow from $\sigma(A+c) = \sigma(A) + c$. $\square$

We end this chapter by pointing out that a similar result as in Theorem 1.2.5(c) can also be proved for matrix semigroups if $e^{tA}$ is strongly positive for every $t > 0$. We do not discuss this further at this point, but a result in the next chapter, Theorem 2.3.1, will contain this as a special case.

# Exercises for Chapter 1

**Exercise 1.1.** Let $A, B \in \mathbb{C}^{n\times n}$.

(a) Give an example to show that $e^{A+B} = e^A e^B$ does not imply $AB = BA$.

(b) If there exists $\varepsilon > 0$ such that $e^{t(A+B)} = e^{tA}e^{tB}$ for all $t \in [0,\varepsilon)$, then show $AB = BA$.

**Exercise 1.2** (Continuation of Theorem 1.3.8)**.** Let $A \in \mathbb{R}^{n\times n}$ be given. Prove that the following are equivalent:

(iv) All off-diagonal entries of $A$ are in $[0,\infty)$, i.e. $A_{jk} \geq 0$ for all indices $j \neq k$.

(v) The matrix $A$ satisfies the *positive minimum principle*, i.e. for all $u \in \mathbb{R}^n_+$ and all $k \in \{1,\dots,n\}$ with $u_k = 0$ one has $(Au)_k \geq 0$.

(vi) The matrix $A$ is *cross positive*, i.e. for all $u, v \in \mathbb{R}^n_+$ with $u^{\mathrm{T}} v = 0$ one has $u^{\mathrm{T}} A v \geq 0$.

(vii) The matrix $A$ satisfies the *Beurling–Deny criterion*, i.e. for every $u \in \mathbb{R}^n$ one has $(u^-)^{\mathrm{T}} A u^+ \geq 0$, where

$$(u^+)_k := \begin{cases} u_k & \text{if } u_k \geq 0, \\ 0 & \text{if } u_k < 0 \end{cases}$$

for all $k \in \{1,\dots,n\}$, and where $u^- := (-u)^+$.

(viii) The matrix $A$ satisfies the *Arendt–Kato inequality*, i.e. for all $u \in \mathbb{R}^n$ and all indices $k$ with $u_k \geq 0$ one has $(Au^+)_k \geq (Au)_k$.

**Exercise 1.3.** The koala (*Phascolarctos cinereus*) is a notoriously lazy animal, sleeping up to 20 hours a day. It is also a very picky eater. Suppose that a particular koala has 3 favourite eucalyptus trees, arranged as in Figure 1.3.1.

For $i, j \in \{1,2,3\}$, let $P_{ij}$ denote the probability that the koala will eat at tree $i$ the following day given that it has eaten at tree $j$ today. Consider the following model:

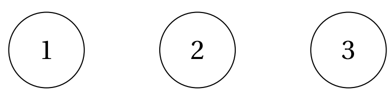


Figure 1.3.1: Eucalyptus trees in an Australian forest (some imagination is required).

- With probability $q \in (0,1)$, the koala will stay at the same tree the following day.
- Since it is lazy, the koala will only move to adjacent trees. Hence, if it has eaten at tree 2 on one day, it will move to either tree 1 or 3 the next day with equal probability (or otherwise stay in place). On the other hand, if it has eaten at tree 1 or 3, it will only move to tree 2 (or otherwise stay in place).

(a) For the probabilities $P_{ij}$ described above, write down the matrix $P = (P_{ij})_{1\le i,j\le 3}$, which is called the **transition matrix** of the model, in terms of $q$. Explain what the $(i,j)$-th entry of the matrix powers $P^k$ represents.

(b) Show that $\mathrm{r}(P) = 1$ and that 1 is a strictly dominant eigenvalue.

(c) In the long run, what can you say about the proportion of days the koala spends at each tree?

**Exercise 1.4.** Let $\lambda_0 \in \mathbb{C}$ and let $J_0 \in \mathbb{C}^{n_0\times n_0}$ denote the Jordan block

$$J_0 = \begin{pmatrix} \lambda_0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_0 \end{pmatrix}.$$

(a) Find and prove a formula for the matrix $J_0^k$ for every $k \in \mathbb{N}_0$.

(b) Find and prove a formula for the matrix $e^{tJ_0}$ for every $t \in [0,\infty)$.

*Hint:* First consider the case $\lambda_0 = 0$ and then use Proposition 1.3.2(d).

(c) Let $A \in \mathbb{C}^{n\times n}$ such that $\mathrm{r}(A) > 0$. Use the Jordan normal form of $A$ to show that there exists a number $c > 0$ such that $\|A^k\| \le c\,\mathrm{r}(A)^k(1 + k^{n-1})$ for each $k \in \mathbb{N}_0$.

(d) Let $A \in \mathbb{C}^{n\times n}$. Use the Jordan normal form of $A$ to show that there exists a number $c > 0$ such that $\|e^{tA}\| \le ce^{t\,\mathrm{s}(A)}(1 + t^{n-1})$ for each $t \in [0,\infty)$.

# Notes for Chapter 1

## Positivity versus non-negativity

As promised in Remark 1.1.2, we now discuss the terminology *positive* and the related question of whether 0 is considered positive in a bit more detail.

**Real numbers:** It is remarkable that even for real numbers, the meaning of the term *positive* depends on the language. The convention that positivity of a real number $\alpha$ means $\alpha > 0$ – while the property $\alpha \geq 0$ is often referred to as $\alpha$ being *non-negative* – is common in English and, for instance, also in German. On the other hand, in French the adjective *positif* typically refers to a number $\alpha \geq 0$.

As the real numbers are defined as an ordered field with a number of additional properties, it is worthwhile to also take a brief look at conventions in the theory of ordered groups and fields. Unsurprisingly, the French meaning of 'positive' is employed by Bourbaki in its definition of ordered groups [Bou07, p. A VI.4]. The same convention is then used in the English translation [Bou03, p. A VI.4].

**Linear algebra:** A substantial amount of literature studies order properties of $\mathbb{R}^n$ and matrices, in particular in relation to the Perron–Frobenius theorem and its applications. In this field, it seems to be most common to call a vector $x$ *positive* if $x_k > 0$ for all indices $k$ (note that we call this property *strongly positive* in Definition 1.2.3). Vectors $x \geq 0$ are typically referred to as *non-negative* vectors in this part of the literature. This terminology has the advantage that it is consistent with the standard conventions for real numbers in English.

Care must be taken, though, since the coordinate-wise relation $\leq$ defines only a partial order on $\mathbb{R}^n$ when $n \geq 2$: there exist vectors $x \in \mathbb{R}^n$ that satisfy neither $x \geq 0$ nor $x \leq 0$. If one gives in to the temptation to call a vector $x \in \mathbb{R}^n$ *negative* if $-x$ is positive, then the common terminology in linear algebra leads to the following situation: $x$ is negative if and only if $x_k < 0$ for all indices $k$ and thus we have a linguistically unpleasant situation where the assertion "$x$ is non-negative" is inequivalent to "$x$ is not negative".

**Real-valued functions:** Some parts of the literature adopt a convention similar to the one in linear algebra: a function $f : \Omega \to \mathbb{R}$ defined on a set $\Omega$ is called *non-negative* if $f(\omega) \geq 0$ for all $\omega \in \Omega$ or, for instance in the setting of $L^p$-spaces, for almost all $\omega \in \Omega$. Accordingly, $f$ is then called *positive* if $f(\omega) > 0$ for (almost) all $\omega \in \Omega$.

This adaptation of the finite-dimensional perspective comes with an additional caveat in infinite dimensions that only becomes apparent when one develops a systematic theory of ordered spaces in infinite dimensions. The property $f(\omega) > 0$ for (almost) all $\omega \in \Omega$ has very different consequences depending on the surrounding space. For example, in $\mathrm{C}(K)$ it implies that $f$ dominates a strictly positive constant on $K$, whereas in $L^p(\Omega)$ it does not. We elaborate on this later in Chapter 6, when we have enough Banach lattice theory available.

**Elements of ordered vector spaces and Banach lattices:** In the theory of ordered vector spaces and Banach lattices, which we introduce in Chapter 6, it is common to call a vector $x$ *positive* if $x \geq 0$; in particular, the zero vector is positive. We follow this convention since we frequently use Banach lattice theory later on. To maintain consistency throughout these notes, we have adopted the same convention in the finite-dimensional setting, as can be seen in Definition 1.1.1.

## Perron–Frobenius and friends

The story of the Perron–Frobenius theorem, and the theory of non-negative matrices[5] in general, has a surprising beginning. At the turn of the 20th century, at the University of Munich (LMU), Oskar Perron was studying the problem of convergence of continued fraction algorithms (German: *Kettenbruchalgorithmen*), following the work of his colleague Alfred Pringsheim. His breakthrough was to reduce the problem to a study of the eigenvalue equation for specific matrices with positive entries (although he did not use this terminology). Consequently Perron was able to simplify the convergence criteria from earlier works (such as those of Pringsheim), and moreover, his methods could be extended to treat the more general case of Jacobi algorithms. This became the subject of his *Habilitation* paper [Per07a], published in *Mathematische Annalen* in 1907.

Clearly Perron recognised the utility of his methods beyond their original purpose and the potential for a systematic theory, for he then followed up with the article *Zur Theorie der Matrices*, which was also published in the *Mathematische Annalen* [Per07b]. In this work, Perron's main theorem corresponds more or less to Theorem 1.2.5(c) in this chapter. Moreover, he could derive the same conclusions under the weaker assumption that $A \geq 0$ and $A^k$ is positive (in our terminology, strongly positive) for some $k \in \mathbb{N}$. However, he expressed dissatisfaction with his rather convoluted argument to achieve this generalisation, and in addition he left open the possibility that a larger class of non-negative matrices could satisfy the conclusions of his theorem.

At this point, Frobenius enters the story. In a series of three papers [Fro08, Fro09, Fro12], he manages to resolve the issues raised by Perron. In a 1908 paper and its sequel in 1909, he proves strengthened versions of Perron's results for positive matrices

[5] In this historical account, we use the classical terminology from linear algebra as explained at the beginning of these notes.

using thoroughly linear-algebraic techniques, especially the properties of determinants. Then, in the 1912 paper, Frobenius extends his investigations to encompass irreducible non-negative matrices. Here, *irreducible* refers to matrices that cannot be put into block upper-triangular form via simultaneous row or column permutations. We encourage readers who are interested in further details (both mathematical and biographical) of the history of the Perron-Frobenius theorem to consult the article [Haw08].

The study of non-negative matrices, stemming from the ideas of Perron and Frobenius, has proved to be very fruitful, and has found diverse applications in the natural and social sciences, from population models (e.g. Leslie matrices) to queuing theory, to input-output models in economics (e.g. the Leontief model), to Google's PageRank algorithm. Thus, the literature on 'Perron–Frobenius theory' is vast. Two texts that are now considered classical include [Sen06], which deals with applications to probability theory and in particular Markov chains, and [BP94], which is notable for its systematic study of positivity with respect to general cones in $\mathbb{R}^n$.

Finally, we cannot omit a mention of the fairly recent monograph [BKFR17], which was based on the material of the 17th Internet Seminar (2013–2014). Part I of that book contains an accessible exposition of the Perron–Frobenius theorem (in particular, Frobenius' contributions), properties of (positive) matrix exponential functions, and numerous applications.

# Chapter 2

# Eventual positivity in finite dimensions

In this chapter, we encounter our main protagonist, eventual positivity (Section 2.2). We shall see that many spectral properties of positive matrix semigroups remain true for eventually positive ones, but to prove this, we first need more advanced tools from spectral theory (Section 2.1). Remarkably though, already in the finite dimensional setting, there are significant differences between the positive and the eventually positive case. The characterisation of eventual positivity of a matrix semigroup $(e^{tA})_{t\geq 0}$ in terms of $A$ (Section 2.3) has a different flavour than for positive semigroups, and the differences become even clearer when it comes to perturbation theory (Section 2.4).

## 2.1 Prelude: Spectral decomposition of matrices

For the analysis of matrix powers and exponentials in Exercise 1.4, you have already encountered a very useful tool: the Jordan normal form. In this section, the same tool is used to derive a variety of spectral properties, so let us fix the notation for it.

Let $A \in \mathbb{C}^{n\times n}$. By a coordinate transform, one can achieve that $A$ is in **Jordan normal form**. This means that $A$ can be written in block diagonal form as

$$A = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix} \tag{2.1.1}$$

for matrices $J_k \in \mathbb{C}^{n_k \times n_k}$ that are called **Jordan blocks**, i.e. each of them has the form

$$J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix} \tag{2.1.2}$$

for a number $\lambda_k \in \mathbb{C}$. The numbers $\lambda_1, \dots, \lambda_m$ are the eigenvalues of $A$, where the number of occurrences of each eigenvalue in this list coincides with its geometric multiplicity. In other words, the geometric multiplicity of $\lambda_k$ is the number of Jordan blocks associated to $\lambda_k$. On the other hand, the algebraic multiplicity of $\lambda_k$ is the sum of the sizes of all Jordan blocks corresponding to $\lambda_k$. We set

$$N_k := J_k - \lambda_k = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix},$$

which is nilpotent of index $n_k$.

For an open set $\emptyset \neq \Omega \subseteq \mathbb{C}$, a point $\lambda \in \Omega$, and an analytic function $f\colon \Omega \setminus \{\lambda\} \to \mathbb{C}^{m \times n}$, the point $\lambda$ is called a **pole** of $f$ if for some $p \in \mathbb{N}$, the $\lim_{\mu \to \lambda} (\mu - \lambda)^p f(\mu)$ exists and is non-zero. In this case, $p$ is unique and is called the **pole order** of $\lambda$. By considering the entries of $f$, one can see that a similar Laurent series expansion as in the scalar-valued case continues to hold.

**Proposition 2.1.1** (Eigenvalues are poles of the resolvent)**.** *Let $\lambda \in \mathbb{C}$ be an eigenvalue of a matrix $A \in \mathbb{C}^{n \times n}$. If the largest Jordan block of $A$ corresponding to $\lambda$ has size $p \geq 1$, then*

(a) *$\lambda$ is a pole of the resolvent $\mathcal{R}(\cdot, A)$ with pole order $p$, i.e. there exist matrices $Q_j \in \mathbb{C}^{n \times n}$ for $j \geq -p+1$ such that $Q_{-p+1} \neq 0$ and*

$$(\mu - A)^{-1} = \sum_{j=-p}^{\infty} Q_{j+1} (\mu - \lambda)^j$$

*for all $\mu \neq \lambda$ that are sufficiently close to $\lambda$.*

(b) *The range of $Q_{-p+1}$ is contained in the eigenspace* $\ker(\lambda - A)$.

*Proof.* (a) We may assume that $A$ is in Jordan normal form (2.1.1). Thus, in order to compute the inverse $(\mu - A)^{-1}$, we only need to compute the inverses $(\mu - J_k)^{-1}$ of the Jordan blocks $J_k \in \mathbb{C}^{n_k \times n_k}$ of $A$. Due to the Neumann series representation of the resolvent from Proposition 1.2.7(b), we obtain for all $\mu \in \rho(A)$ and all indices $k$,

$$(\mu - J_k)^{-1} = (\mu - \lambda_k)^{-1} \left(\mathrm{id} - (\mu - \lambda_k)^{-1} N_k\right)^{-1} = \sum_{j=0}^{n_k - 1} \frac{N_k^j}{(\mu - \lambda_k)^{j+1}}. \tag{2.1.3}$$

For $\lambda \neq \lambda_k$, the series in (2.1.3) is holomorphic near $\lambda$, so one gets a Taylor expansion of $(\mu - J_k)^{-1}$ about the point $\lambda$. In particular, only blocks with $\lambda = \lambda_k$ can produce poles at $\lambda$ the formula for $(\mu - J_k)^{-1}$ immediately gives the coefficients of the Laurent series expansion of $(\mu - J_k)^{-1}$ about the point $\lambda$ and contributes a pole of order $n_k$. Thus, $\lambda$ is a pole of order $\max_k n_k = p$.

We infer from (2.1.3) that the coefficient of $(\mu - \lambda)^{-p}$ in $(\mu - J_k)^{-1}$ is $N_k^{p-1} \neq 0$ whenever $n_k = p$ and $0$ if $n_k < p$ or $\lambda \neq \lambda_k$. Hence, the blocks in $Q_{-p+1}$ corresponding to $n_k = p$ and $\lambda_k = \lambda$ are non-zero. In particular, $Q_{-p+1} \neq 0$.

(b) For each index $k$, $\operatorname{rg}(N_k^{p-1}) \subseteq \ker(\lambda_k - J_k)$ because

$$(\lambda_k - J_k)N_k^{p-1} = -N_k^p = 0.$$

From the proof of (a), we know that the non-zero blocks in $Q_{-p+1}$ are exactly those for which $\lambda_k = \lambda$ and $n_k = p$. Consequently, $\operatorname{rg} Q_{-p+1}$ is contained in $\ker(\lambda - A)$. □

The formula (2.1.3) has a number of useful consequences that we discuss now. We need the following crucial observation about contour integration in complex analysis.

**Proposition 2.1.2.** *Let $\lambda \in \mathbb{C}$ and let $\gamma$ be a closed $\mathrm{C}^1$-path in $\mathbb{C} \setminus \{\lambda\}$.*

(a) *For each integer $j \neq 1$ one has $\oint_\gamma \frac{1}{(\mu-\lambda)^j} \,\mathrm{d}\mu = 0$.*

(b) *If $\gamma$ encircles $\lambda$ precisely once, then $\frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{1}{\mu-\lambda} \,\mathrm{d}\mu = 1$.*

(c) *If $\gamma$ does not encircle $\lambda$, then $\oint_\gamma \frac{1}{\mu-\lambda} \,\mathrm{d}\mu = 0$.*

Proposition 2.1.2 can be applied to every entry of the $(\mu - J_k)^{-1}$ in formula (2.1.3). Doing this for all the Jordan blocks of $A$, one immediately obtains the following.

**Corollary 2.1.3.** *Let $A \in \mathbb{C}^{n\times n}$ and let $\gamma$ be a closed $\mathrm{C}^1$-path in $\mathbb{C} \setminus \sigma(A)$.*

(a) *If $\gamma$ encircles each eigenvalue of $A$ precisely once, then $\frac{1}{2\pi \mathrm{i}} \oint_\gamma (\mu - A)^{-1} \,\mathrm{d}\mu = \mathrm{id}$.*

(b) *If $\gamma$ does not encircle any eigenvalue of $A$, then $\oint_\gamma (\mu - A)^{-1} \,\mathrm{d}\mu = 0$.*

Corollary 2.1.3 now comes in handy to prove the representation formula, and hence the uniqueness, in the following proposition.

**Proposition 2.1.4** (Spectral decomposition)**.** *Let $A \in \mathbb{C}^{n\times n}$ and let $\sigma_0 \subseteq \sigma(A)$. There exists precisely one projection $P \in \mathbb{C}^{n\times n}$ that has the following properties:*

(a) *$P$ commutes with $A$.*

(b) *The restrictions of $A$ to the range and the kernel of $P$ have the spectra*

$$\sigma\left(A|_{\operatorname{rg} P}\right) = \sigma_0 \qquad \text{and} \qquad \sigma\left(A|_{\ker P}\right) = \sigma(A) \setminus \sigma_0.$$

*Moreover, $P$ is given by the formula*

$$P = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (\mu - A)^{-1} \,\mathrm{d}\mu \tag{2.1.4}$$

*for any closed $\mathrm{C}^1$-path $\gamma$ in $\mathbb{C}$ that encircles each element of $\sigma_0$ precisely once, but no element of $\sigma(A) \setminus \sigma_0$.*

*Proof. Existence:* After a coordinate transform, we may assume that $A$ is in Jordan normal form, i.e. that it is given by the formula (2.1.1); we use the notation specified next to this formula. After ordering the eigenvalues $\lambda_1, \dots, \lambda_m$ appropriately, we can find an $\ell \in \{0, 1, \dots, m\}$ such that $\sigma_0 = \{\lambda_k : 1 \le k \le \ell\}$ and $\sigma(A) \setminus \sigma_0 = \{\lambda_k : \ell < k \le m\}$; we allow $\ell = 0$ to account for the case $\sigma_0 = \emptyset$. Now, let $P \in \mathbb{C}^{n \times n}$ be the projection onto the first $n_1 + \dots + n_\ell$ components of $\mathbb{C}^n$. Then $P$ has the required properties.

*Uniqueness and integral formula:* Let $P \in \mathbb{C}^{n \times n}$ be a projection that satisfies (a) and (b) and let the complex path $\gamma$ have the properties specified at the end of the proposition. It suffices to show that $P$ is given by the claimed path integral.

Since $P$ commutes with $A$, it also commutes with $(\mu - A)^{-1}$ for every $\mu \in \rho(A)$ and one has $(\mu - A)^{-1}|_{\operatorname{rg} P} = (\mu - A|_{\operatorname{rg} P})^{-1}$ for all such $\mu$; the same also holds for the restriction to $\ker P$. So Corollary 2.1.3(a) applied to $A|_{\operatorname{rg} P}$

$$\frac{1}{2\pi \mathrm{i}} \oint_\gamma (\mu - A)^{-1} \,\mathrm{d}\mu|_{\operatorname{rg} P} = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (\mu - A|_{\operatorname{rg} P})^{-1} \,\mathrm{d}\mu = \mathrm{id}_{\operatorname{rg} P} = P|_{\operatorname{rg} P}$$

and similarly, Corollary 2.1.3(b) applied to $A|_{\ker P}$ gives

$$\frac{1}{2\pi \mathrm{i}} \oint_\gamma (\mu - A)^{-1} \,\mathrm{d}\mu|_{\ker P} = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (\mu - A|_{\ker P})^{-1} \,\mathrm{d}\mu = 0 = P|_{\ker P}.$$

This shows the claimed formula for $P$. □

**Definition 2.1.5** (Spectral projections)**.** In the situation of Proposition 2.1.4, the projection $P$ is called the **spectral projection** of $A$ associated to $\sigma_0$.

In the situation of Proposition 2.1.4, note that the complementary projection $1 - P$ also commutes with $A$ and satisfies $\operatorname{rg}(1 - P) = \ker P$ and $\ker(1 - P) = \operatorname{rg} P$. Hence, it follows that $1 - P$ is the spectral projection of $A$ associated to $\sigma(A) \setminus \sigma_0$.

Recall that an eigenvalue $\lambda$ of a square matrix $A$ is called **semisimple** if its geometric and algebraic multiplicities coincide.

**Proposition 2.1.6.** *Let $A \in \mathbb{C}^{n \times n}$ and let $P$ be the spectral projection of $A$ associated to an eigenvalue $\lambda \in \sigma(A)$.*[1]

(a) *$P$ is equal to the coefficient $Q_0$ of the term $(\mu - \lambda)^{-1}$ in the Laurent series expansion of $(\mu - A)^{-1}$ in Proposition 2.1.1(a).*

(b) *$\operatorname{rg} P = \bigcup_{k=1}^n \ker(\lambda - A)^k$, i.e. the range of $P$ coincides with the generalised eigenspace of $\lambda$. In particular, $\dim \operatorname{rg} P$ is the algebraic multiplicity of the eigenvalue $\lambda$.*

(c) *The following are equivalent:*

  (i) *The eigenvalue $\lambda$ is a first order pole of the resolvent $\mathcal{R}(\cdot, A)$.*

[1] This is an informal way of saying that $P$ is the spectral projection associated to the set $\{\lambda\}$.

(ii) *The eigenvalue $\lambda$ is a first order pole of the dual resolvent $\mathcal{R}(\cdot, A^{\mathrm{T}})$.*

(iii) *The limit $\lim_{\mu\to\lambda}(\mu-\lambda)\mathcal{R}(\mu, A)$ exists.*

(iv) *The eigenvalue $\lambda$ is semisimple.*

(v) *The range of $P$ consists of eigenvectors, i.e. $\operatorname{rg} P = \ker(\lambda - A)$.*

*If the equivalent conditions (i)–(v) are satisfied, then $\lim_{\mu\to\lambda}(\mu-\lambda)\mathcal{R}(\mu, A) = P$.*

(d) *If $\ker(\lambda - A) = \operatorname{span}\{u\}$ and $v \in \ker(\lambda - A^{\mathrm{T}})$ satisfy $v^{\mathrm{T}} u = 1$, then $P = uv^{\mathrm{T}}$.*

*Proof.* (a) This follows from the integral representation of $P$ in Proposition 2.1.4 and from the integration formula in Proposition 2.1.2(a).

(b) This follows from how $P$ was constructed in the existence argument in the proof of Proposition 2.1.4.

(c) If $\lambda$ is a first order pole, then $\lim_{\mu\to\lambda}(\mu-\lambda)\mathcal{R}(\mu, A) = Q_0$ by the Laurent series expansion of $\mathcal{R}(\cdot, A)$ about $\lambda$ (Proposition 2.1.1(a)). Moreover, $Q_0 = P$ according to (a). Let us now prove the claimed equivalence.

"(i) $\Leftrightarrow$ (ii)": One has $\mathcal{R}(\mu, A^{\mathrm{T}}) = \mathcal{R}(\mu, A)^{\mathrm{T}}$ for all $\mu \in \rho(A^{\mathrm{T}}) = \rho(A)$, so one can see the claimed equivalence by taking the transposition operation out of the Laurent series expansion in Proposition 2.1.1(a).

"(i) $\Leftrightarrow$ (iii)": For a scalar-valued holomorphic function $f$ that has an isolated singularity at $\lambda$ it is a standard result from complex analysis that $\lambda$ is a first order pole of $f$ if and only if $\lim_{\mu\to\lambda}(\mu-\lambda)f(\mu)$ exists. The claim now follows from applying this fact to all components of the resolvent.

"(i) $\Leftrightarrow$ (iv)": Since $\lambda$ is semisimple if and only if every Jordan block corresponding to $\lambda$ has size 1, which is equivalent to $\lambda$ having pole order one by Proposition 2.1.1.

"(i) $\Rightarrow$ (v)": Note that $\ker(\lambda - A) \subseteq \operatorname{rg} P = \operatorname{rg} Q_0$ by (b) and (a). If the pole order at $\lambda$ is 1, the converse inclusion $\operatorname{rg} Q_0 \subseteq \ker(\lambda - A)$ follows from Proposition 2.1.1(b).

"(v) $\Rightarrow$ (iv)": The geometric multiplicity of $\lambda$ is $\dim\ker(\lambda - A)$ and the algebraic multiplicity of $\lambda$ is $\dim\operatorname{rg} P$, according to (b). It follows from (v) that the two are equal.

(d) The assumptions ensure that $\lambda$ is even algebraically simple (hence, semisimple) due to Lemma 1.2.8. Now (c) implies $\operatorname{rg} P$ is spanned by $u$. By applying the same argument to $A^{\mathrm{T}}$ – whose spectral projection for the eigenvalue $\lambda$ is $P^{\mathrm{T}}$ due to formula (2.1.4) – we also see that $\operatorname{rg} P^{\mathrm{T}}$ is spanned by $v$. Thus, $P = \alpha uv^{\mathrm{T}}$ for a scalar $\alpha$. As $P$ is a projection and $v^{\mathrm{T}} u = 1$, it follows that $\alpha = 1$. $\square$

**Theorem 2.1.7** (Spectral mapping theorem for the matrix exponential function)**.** *Let $A \in \mathbb{C}^{n\times n}$ and let $t \in \mathbb{R}$. Then one has $\sigma(e^{tA}) = e^{t\sigma(A)}$. More precisely:*

(a) *If $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ with eigenvector $z \in \mathbb{C}^n$, then $e^{t\lambda}$ is an eigenvalue of $e^{tA}$ with eigenvector $z$.*

(b) *If $\mu \in \sigma(e^{tA})$, then there exists $\lambda \in \sigma(A)$ such that $\mu = e^{t\lambda}$.*

*Proof.* (a) It follows from $Az = \lambda z$ that $(tA)^j z = (t\lambda)^j z$ for all integer $j \geq 0$, hence

$$e^{tA} z = \sum_{j=0}^{\infty} \frac{(tA)^j}{j!} z = \sum_{j=0}^{\infty} \frac{(t\lambda)^j}{j!} z = e^{t\lambda} z.$$

(b) After a coordinate transformation, we may assume $A$ is in Jordan normal form (2.1.1). Then by Exercise 1.4(b), $e^{tA}$ is an upper triangular matrix whose eigenvalues are $e^{t\lambda_1}, \dots, e^{t\lambda_m}$, where $\lambda_1, \dots, \lambda_m$ are the eigenvalues of $A$ (counted with their geometric multiplicity). Hence $\mu = e^{t\lambda}$ for some $\lambda \in \sigma(A)$. □

## 2.2 Eventually positive matrix semigroups

Our main objects of study for the rest of Chapter 2 are matrix semigroups $(e^{tA})_{t \geq 0}$ which are positive for all sufficiently large times.

**Definition 2.2.1** (Eventually (strongly) positive matrix semigroups)**.** Let $A \in \mathbb{R}^{n \times n}$.

(a) The matrix semigroup $(e^{tA})_{t \geq 0}$ is called **eventually positive** if there exists $t_0 \geq 0$ such that $e^{tA} \geq 0$ for all $t \in [t_0, \infty)$.

(b) The matrix semigroup $(e^{tA})_{t \geq 0}$ is called **eventually strongly positive** if there exists $t_0 \geq 0$ such that $e^{tA} x \succeq \mathbb{1}$ for all $0 \neq x \in \mathbb{R}^n_+$ and all $t \in [t_0, \infty)$.

This definition uses Notation 1.2.4 again, i.e. for a given $t$, the inequality $e^{tA} x \succeq \mathbb{1}$ means that there exists a number $c > 0$ such that $e^{tA} x \geq c\,\mathbb{1}$. Observe that $c$ can a priori depend on $t$.

**Examples 2.2.2.**

(a) The matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is nilpotent and satisfies $A^k = 0$ for all $k \geq 3$. Moreover,

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{so} \qquad e^{tA} = \begin{pmatrix} 1 & t & \frac{t^2}{2} - t \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

for all $t \geq 0$. Thus, $(e^{tA})_{t \geq 0}$ is eventually positive but not eventually strongly positive. Since $\frac{t^2}{2} - t < 0$ for $t \in (0,2)$, the semigroup is not positive. Alternatively, this follows from the characterisation of positive semigroups in Theorem 1.3.8 since the off-diagonal entry $A_{13} = -1$ is strictly negative.

(b) Consider the matrix

$$A := U\begin{pmatrix} 0 & & \\ & -1 & \\ & & -9 \end{pmatrix} U^{\mathrm{T}} = \begin{pmatrix} -2 & -1 & 3 \\ -1 & -2 & 3 \\ 3 & 3 & -6 \end{pmatrix},$$

where $U := (u_1, u_2, u_3) \in \mathbb{R}^{3\times 3}$ is the orthogonal matrix with the columns

$$u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}, \quad u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^{\mathrm{T}}, \quad u_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1 & -2 \end{pmatrix}^{\mathrm{T}}.$$

Let $P \in \mathbb{R}^{3\times 3}$ be the projection onto the first component of $\mathbb{R}^3$. Then $e^{tA} \to UPU^{\mathrm{T}} = u_1 u_1^{\mathrm{T}}$ as $t \to \infty$. Since every entry of $u_1 u_1^{\mathrm{T}}$ is $\frac{1}{3}$, it follows that $(e^{tA})_{t\geq 0}$ is eventually strongly positive. However as $A$ has some negative off-diagonal entries, so by the characterisation of positive semigroups in Theorem 1.3.8, $(e^{tA})_{t\geq 0}$ is not positive.

Parts (a) and (b) of the following result show that Theorem 1.3.9 about positive semigroups continues to hold in the eventually positive case. We also add a third property (c) which we will use to study perturbation theory in Theorem 2.4.2.

**Theorem 2.2.3** (Perron–Frobenius for eventually positive matrix semigroups)**.** *Let $A \in \mathbb{R}^{n\times n}$ be such that $(e^{tA})_{t\geq 0}$ is eventually positive. The following assertions hold:*

(a) *The spectral bound $\mathrm{s}(A)$ is an eigenvalue of $A$ with an eigenvector $x \geq 0$.*

(b) $\mathrm{s}(A)$ *is a strictly dominant eigenvalue of $A$, i.e.* $\operatorname{Re}\lambda < \mathrm{s}(A)$ *for all* $\lambda \in \sigma(A) \setminus \{\mathrm{s}(A)\}$.

(c) *If* $\mathrm{s}(A)$ *is semisimple, then its associated spectral projection is positive.*

*Proof.* By replacing $A$ with $A - \mathrm{s}(A)$ (which does not affect questions of positivity), we may assume that $\mathrm{s}(A) = 0$. Let $t_0 \in [0,\infty)$ be such that $e^{tA} \geq 0$ for all $t \geq t_0$.

(a) and (b) We first prove that 0 is an eigenvalue of $A$ and that there are no non-zero eigenvalues of $A$ on the imaginary line. Since $\mathrm{s}(A) = 0$, it follows from the spectral mapping theorem for the matrix exponential function (Theorem 2.1.7) that $e^{tA}$ has spectral radius 1 for each $t \geq 0$. Let $i\beta \in i\mathbb{R}$ be an eigenvalue of $A$. Again by Theorem 2.1.7, $e^{i\beta t}$ is an eigenvalue of $e^{tA}$ for each $t \in [0,\infty)$. We need to show that $\beta = 0$.

For each index $j \in \{1,\dots,n\}$ one has $(e^{0A})_{jj} = 1$ hence by the uniqueness theorem for analytic functions, the set $\bigcup_j \left\{t \in [0,\infty) : (e^{tA})_{jj} = 0\right\}$ does not accumulate in $[0,\infty)$. Thus, there exist times $t_2 > t_1 \geq t_0$ such that for each $t \in [t_1, t_2]$, the diagonal entries of $e^{tA}$ are strictly positive. According to Theorem 1.2.5(b) this implies, for each such $t$, that the spectral radius 1 is a radially strictly dominant eigenvalue of $e^{tA}$, i.e. the matrix $e^{tA}$ does not have eigenvalues on the unit circle except for the number 1. Thus, $e^{i\beta t} = 1$ for all $t \in [t_1, t_2]$, which implies that $\beta = 0$, as claimed.

Now we show the existence of a positive eigenvector of $A$ for the eigenvalue 0. With the notation of the Laurent series expansion of $(\mu - A)^{-1}$ about the eigenvalue 0 in

Proposition 2.1.1(a), one gets $Q_{-p+1} = \lim_{\mu\to 0} \mu^p(\mu - A)^{-1}$. Using the Laplace transform representation of the resolvent (Lemma 1.3.7) yields

$$Q_{-p+1} = \lim_{\mu\downarrow 0}\Big(\underbrace{\mu^p\int_0^{t_0} e^{-t\mu}e^{tA}\,\mathrm{d}t}_{\to 0} + \underbrace{\mu^p\int_{t_0}^{\infty} e^{-t\mu}e^{tA}\,\mathrm{d}t}_{\geq 0}\Big) \geq 0.$$

Since $Q_{-p+1}$ is non-zero and $\mathbb{R}^n_+$ spans $\mathbb{R}^n$, we can find a vector $x \in \mathbb{R}^n_+$ such that $0 \leq Q_{-p+1}x \neq 0$. According to Proposition 2.1.1(b) that range of $Q_{-p+1}$ is contained in $\ker A$, so $Q_{-p+1}x$ is indeed a positive eigenvector of $A$ for the eigenvalue $\mathrm{s}(A) = 0$.

(c) Due to semisimplicity, Proposition 2.1.6(a) and (c) give $P = Q_0 = Q_{-p+1} \geq 0$. □

**Example 2.2.4.** In Theorem 2.2.3(c), the spectral projection can fail to be positive if $\mathrm{s}(A)$ is not a semisimple eigenvalue. Indeed, let

$$A := T\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}T^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{for} \quad T := \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & -2 \\ 1 & 0 & -1 \end{pmatrix}$$

Since all off-diagonal entries of $A$ are $\geq 0$, the semigroup $(e^{tA})_{t\geq 0}$ is positive (Theorem 1.3.8). The given Jordan normal form of $A$ shows that $\mathrm{s}(A) = 0$ is not semisimple. The spectral projection of $A$ associated to the eigenvalue 0 is

$$P = T\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}T^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & -1 & 4 \end{pmatrix} \ngeq 0.$$

## 2.3 Characterisation

By Theorem 1.3.8, positivity of $(e^{tA})_{t\geq 0}$ is equivalent to the positivity of $\mathcal{R}(\lambda, A)$ for all $\lambda \in (\mathrm{s}(A), \infty)$. Parts (i) and (ii) of the next theorem provide a related characterisation for eventual strong positivity. Parts (iii) and (iv) show that eventual strong positivity can be characterised in terms of Perron–Frobenius like spectral properties. In this sense, Perron–Frobenius theory is closer to eventual positivity than to positivity.

**Theorem 2.3.1.** *Let $A \in \mathbb{R}^{n\times n}$. The following are equivalent:*

(i) $(e^{tA})_{t\geq 0}$ *is eventually strongly positive.*

(ii) $\mathrm{s}(A)$ *is a strictly dominant eigenvalue of $A$ and there exists $\lambda_0 > \mathrm{s}(A)$ such that $\mathcal{R}(\lambda, A)$ is strongly positive for all $\lambda \in (\mathrm{s}(A), \lambda_0)$.*

(iii) $\mathrm{s}(A)$ *is a strictly dominant eigenvalue of $A$ and the associated spectral projection is strongly positive.*

(iv) $\mathrm{s}(A)$ *is a strictly dominant eigenvalue of $A$ and the eigenspaces* $\ker(\mathrm{s}(A) - A)$ *and* $\ker(\mathrm{s}(A) - A^{\mathrm{T}})$ *are spanned by strongly positive vectors.*

*If these equivalent assertions hold, then the eigenvalue* $\mathrm{s}(A)$ *is even algebraically simple.*

*Proof.* As usual, we may assume $\mathrm{s}(A) = 0$. Note that if the spectral projection $P$ associated to 0 is strongly positive, then the eigenvalue 0 of $A$ is algebraically simple (and in turn, semisimple): indeed, if $P$ is strongly positive, then Perron–Frobenius (Theorem 1.2.5(c)) guarantees that $\mathrm{r}(P) > 0$ and $\ker(\mathrm{r}(P) - P)$ is spanned by a strongly positive vector. On the other hand, $\mathrm{r}(P) = 1$, as $P$ is a non-zero projection. Consequently, $\operatorname{rg} P = \ker(1 - P)$ is one-dimensional. As $\dim \operatorname{rg} P$ is the algebraic multiplicity of the eigenvalue $\mathrm{s}(A) = 0$ (Proposition 2.1.6(b)), it follows that 0 is algebraically simple.

"(i) ⇒ (iv)": By Theorem 2.2.3 (Perron–Frobenius for eventually positive matrix semigroups) the spectral bound 0 is a strictly dominant eigenvalue of $A$. Choose $t_0 > 0$ such that the matrix $e^{t_0 A}$ is strongly positive. Due to the spectral mapping theorem for the matrix exponential function (Theorem 2.1.7(a)), one has $\{0\} \neq \ker A \subseteq \ker(1 - e^{t_0 A})$, and the latter space is spanned by a strongly positive vector according to the Perron–Frobenius theorem for strongly positive matrices (Theorem 1.2.5(c)). Thus, $\ker A = \ker(1 - e^{t_0 A})$, which proves the claim for $\ker A$. The same argument applies to $\ker(A^{\mathrm{T}})$, since $e^{t_0 A^{\mathrm{T}}} = (e^{t_0 A})^{\mathrm{T}}$ is also strongly positive.

"(iv) ⇒ (iii)": By assumption, $\ker A$ and $\ker A^{T}$ are spanned by strongly positive vectors $u, v$ respectively. Replacing $u$ by a scalar multiple, we may assume that $v^{\mathrm{T}} u = 1$. Proposition 2.1.6(d) now yields $P = uv^{\mathrm{T}}$ is strongly positive.

"(iii) ⇒ (i)": We have seen that the strong positivity of $P$ implies that 0 is semisimple. This ensures $\operatorname{rg} P = \ker A$ due to Proposition 2.1.6(c). The spectral mapping theorem (Theorem 2.1.7) thus implies that $e^{tA}$ acts as the identity matrix on $\operatorname{rg} P$.

Also, since 0 is a strictly dominant eigenvalue of $A$, all eigenvalues of $A|_{\ker P}$ have strictly negative real part by Proposition 2.1.4(b). Therefore, $e^{tA}|_{\ker P} \to 0$ as $t \to \infty$ according to Proposition 1.3.6. Consequently, $e^{tA} = e^{tA} P + e^{tA}(1 - P) \to P$ as $t \to \infty$. The strong positivity of $P$ hence implies the eventual strong positivity of $(e^{tA})_{t \geq 0}$.

"(ii) ⇒ (iv)": Let $\lambda \in (0, \lambda_0) = (\mathrm{s}(A), \lambda_0)$. One has

$$\sigma\big(\mathcal{R}(\lambda, A)\big) = \left\{ \frac{1}{\lambda - \mu} : \mu \in \sigma(A) \right\}.$$

As $\mathrm{s}(A) = 0 \in \sigma(A)$, it follows that $\mathrm{r}(\mathcal{R}(\lambda, A)) = \frac{1}{\lambda}$ is an eigenvalue of $\mathcal{R}(\lambda, A)$.

Observe that $\ker A = \ker\big(\frac{1}{\lambda} - \mathcal{R}(\lambda, A)\big)$ and the latter space is spanned by a strongly positive vector according to the Perron–Frobenius theorem for strongly positive matrices (Theorem 1.2.5(c)). The same argument can be applied to $A^{\mathrm{T}}$.

"(iii) ⇒ (ii)": Since $P$ is strongly positive, 0 is semisimple as already shown. Proposition 2.1.6(c) thus gives $\lim_{\mu \to 0} \mu \mathcal{R}(\mu, A) = P$. As $P$ is strongly positive, this implies that $\mathcal{R}(\mu, A)$ is also strongly positive for all $\mu > 0$ that are sufficiently close to 0. □

## 2.4 Perturbations

We conclude this chapter with a sneak peek of the perturbation theory for eventually positive semigroups. By **perturbations** – more precisely, additive perturbations – we mean the following: given two matrices $A, B \in \mathbb{C}^{n\times n}$, we study which properties of the semigroup $(e^{tA})_{t\geq 0}$ are inherited by the **perturbed** semigroup $\left(e^{t(A+B)}\right)_{t\geq 0}$ if $B$ has sufficiently nice properties. In other words, $B$ is viewed as a perturbation of $A$ , and our goal is to determine which semigroup properties are preserved under such perturbations.

A simple instance of such a perturbation result is the fact that positive perturbations do not destroy the positivity of a semigroup. This is a particular case of the following.

**Proposition 2.4.1.** *Let $A, B \in \mathbb{R}^{n\times n}$ and assume that $(e^{tA})_{t\geq 0}$ is positive. If all off-diagonal entries of $B$ are $\geq 0$, then the perturbed semigroup $\left(e^{t(A+B)}\right)_{t\geq 0}$ is also positive.*

*Proof.* By Theorem 1.3.8, the semigroup generated by a matrix $C \in \mathbb{R}^{n\times n}$ is positive if and only if all off-diagonal entries of $C$ are $\geq 0$. The assertion is now immediate. □

It is natural to ask whether a similar perturbation result holds for eventually positive semigroups: if $(e^{tA})_{t\geq 0}$ is eventually positive and $B \in \mathbb{R}^{n\times n}$ is a positive matrix, does it follow that the perturbed semigroup $\left(e^{t(A+B)}\right)_{t\geq 0}$ is also eventually positive? The answer to this question is quite surprising (and perhaps disappointing): unless the unperturbed semigroup is already positive, there always exists a positive perturbation that destroys the eventual positivity. We prove this in the following theorem.

**Theorem 2.4.2.** *Let $A \in \mathbb{R}^{n\times n}$. The following are equivalent.*

(i) *For every $B \in \mathbb{R}^{n\times n}_+$, the semigroup $\left(e^{t(A+B)}\right)_{t\geq 0}$ is eventually positive.*

(ii) *For every $B \in \mathbb{R}^{n\times n}_+$ of rank $\leq 1$, the semigroup $\left(e^{t(A+B)}\right)_{t\geq 0}$ is eventually positive.*

(iii) *The semigroup $(e^{tA})_{t\geq 0}$ is positive.*

*Proof.* *"(iii) ⇒ (i)":* In this case, $\left(e^{t(A+B)}\right)_{t\geq 0}$ is even positive (Proposition 2.4.1).

*"(i) ⇒ (ii)":* This is trivial.

*"(ii) ⇒ (iii)":* This is the surprising part. The key ingredient is the Sherman–Morrison–Woodbury formula for rank-one perturbations of resolvents, presented in Exercise 2.3.

As before, we assume $\mathrm{s}(A) = 0$ without loss of generality. According to Theorem 1.3.8, it suffices to prove that $\mathcal{R}(\mu, A) \geq 0$ for all $\mu > 0$. To achieve this, in fact it suffices to prove that $v^{\mathrm{T}}\mathcal{R}(\mu, A) \geq 0$ for all $v \geq \mathbb{1}$ and all $\mu > 0$, since strongly positive vectors are dense in $\mathbb{R}^n_+$. Thus let us fix such a vector $v \geq \mathbb{1}$ and a number $\mu > 0$.

Firstly, observe that assumption (ii) with $B = 0$ implies that $(e^{tA})_{t\geq 0}$ is eventually positive. By Theorem 2.2.3, we deduce that $\mathrm{s}(A) = 0$ is an eigenvalue of $A$ with an eigenvector $u \geq 0$. Thus $v^{\mathrm{T}} u > 0$ and there exists $\alpha > 0$ such that $\alpha v^{\mathrm{T}} u = \mu$.

Consider the rank-one matrix $B := \alpha u v^{\mathrm{T}}$. By Exercise 2.3(b), we have $\mathrm{s}(A+\alpha u v^{\mathrm{T}}) = \mu$, it is a semisimple eigenvalue of $A+B$, and formula (2.4.2) (with $\lambda_0 = 0$) yields

$$(\lambda-\mu)\mathcal{R}(\lambda, A+\alpha u v^{\mathrm{T}}) = (\lambda-\mu)\mathcal{R}(\lambda, A) + \alpha u v^{\mathrm{T}}\mathcal{R}(\lambda, A)$$

for all $\lambda > \mu$. Due to semisimplicity, Proposition 2.1.6(c) ensures that the spectral projection corresponding to the eigenvalue $\mu$ of $A+B$ is given by

$$\lim_{\lambda \downarrow \mu}(\lambda-\mu)\mathcal{R}(\lambda, A+\alpha u v^{\mathrm{T}}) = \alpha u v^{\mathrm{T}}\mathcal{R}(\mu, A).$$

By hypothesis, $A+B$ generates an eventually positive semigroup, so this projection is positive by Theorem 2.2.3(c). As $u \geq 0$ and non-zero, it follows that $v^{\mathrm{T}}\mathcal{R}(\mu, A) \geq 0$. □

Theorem 2.4.2 is not quite the end of the story. The notion "perturbation" already suggests that one is often interested in perturbations that are small in some sense. Furthermore, the above theorem leaves open the possibility that perhaps a more positive result (pun intended) holds for eventual *strong* positivity. As it turns out, one can show that eventual strong positivity of a semigroup $(e^{tA})_{t\geq 0}$ is preserved by all perturbations $B \geq 0$ that are sufficiently small in operator norm. In fact, such a result holds even in infinite dimensions, as we will see in Chapter 13, where perturbation theory for eventually positive semigroups is developed in greater depth.

# Exercises for Chapter 2

**Exercise 2.1.** Consider the matrix

$$A := T \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} T^{-1} \in \mathbb{R}^{4\times 4}$$

for an invertible matrix $T \in \mathbb{R}^{4\times 4}$.

(a) Compute the Laurent series expansion of $\mathcal{R}(\cdot, A)$ about the spectral value 1 and the associated spectral projection. What is the pole order of $\mathcal{R}(\cdot, A)$ at 1?

(b) Compute the Laurent series expansion of $\mathcal{R}(\cdot, A)$ about the spectral value 0 and the associated spectral projection. What is the pole order of $\mathcal{R}(\cdot, A)$ at 0?

Does one have equality of the subspaces in Proposition 2.1.1(b)?

(c) Find a $T$ such that $(e^{tA})_{t\geq 0}$ is eventually strongly positive.

**Exercise 2.2.** Prove the following assertions are equivalent for $A \in \mathbb{R}^{n\times n}$:

(i) $(e^{tA})_{t\geq 0}$ is eventually positive.

(ii) For every $0 \leq x \in \mathbb{R}^n$, there exists $t_0 = t_0(x) \geq 0$ such that $e^{tA}x \geq 0$ for all $t \geq t_0$.

(iii) For every $0 \leq x \in \mathbb{R}^n$ and $0 \leq y \in \mathbb{R}^n$, there exists $t_0 = t_0(x, y) \geq 0$ such that $y^{\mathrm{T}} e^{tA} x \geq 0$ for all $t \geq t_0$.

**Exercise 2.3** (Sherman–Morrison–Woodbury formula)**.** Let $A \in \mathbb{C}^{n\times n}$ and let $u, v \in \mathbb{C}^n$.

(a) If $A$ is invertible, prove that $A - uv^{\mathrm{T}}$ is invertible if and only if $v^{\mathrm{T}} A^{-1} u \neq 1$, and in this case it holds that

$$(A - uv^{\mathrm{T}})^{-1} = A^{-1} + \frac{1}{1 - v^{\mathrm{T}} A^{-1} u} A^{-1} u v^{\mathrm{T}} A^{-1}. \tag{2.4.1}$$

(b) If $\lambda \in \rho(A)$, and $u$ is an eigenvector corresponding to an eigenvalue $\lambda_0 \in \mathbb{C}$ of $A$, deduce that $\lambda - (A + uv^{\mathrm{T}})$ is invertible if and only if $v^{\mathrm{T}} u \neq \lambda - \lambda_0$, and in this case

$$\mathcal{R}(\lambda, A + uv^{\mathrm{T}}) = \mathcal{R}(\lambda, A) + \frac{1}{(\lambda - \lambda_0) - v^{\mathrm{T}} u} uv^{\mathrm{T}} \mathcal{R}(\lambda, A). \tag{2.4.2}$$

Moreover, deduce that $\lambda_0 + v^{\mathrm{T}} u$ is a semisimple eigenvalue of $A + uv^{\mathrm{T}}$.

**Exercise 2.4** (Another characterisation of eventual strong positivity)**.** Let $A \in \mathbb{R}^{n\times n}$.

(a) Assume that there exists $c \in \mathbb{R}$ and $k_0 \in \mathbb{N}$ such that $(A + cI)^k$ is strongly positive for all integers $k \geq k_0$. Show that $(e^{tA})_{t\geq 0}$ is eventually strongly positive.

(b) Suppose $B \in \mathbb{C}^{n\times n}$ is a matrix such that $\mathrm{r}(B) > 0$ is a semisimple and radially strictly dominant eigenvalue (see Theorem 1.2.5(b)). Prove that $\left(\frac{B}{\mathrm{r}(B)}\right)^k$ converges to the spectral projection associated with $\mathrm{r}(B)$ as $k \to \infty$. [*Hint:* Jordan normal form.]

(c) Assume that $(e^{tA})_{t\geq 0}$ is eventually strongly positive. Use part (b) to deduce that there exists $k_0 \in \mathbb{N}$ and $c \in \mathbb{R}$ such that $(A + cI)^k$ is strongly positive for all $k \geq k_0$.

**Exercise 2.5** (Eventual (?) positivity in two dimensions)**.** Let $A \in \mathbb{R}^{2\times 2}$. Show that if $(e^{tA})_{t\geq 0}$ is eventually (strongly) positive, then $(e^{tA})_{t\geq 0}$ is (strongly) positive.

*Hint:* as a first step, think about what $\sigma(A)$ could look like.

# Notes for Chapter 2

## Spectral projections

For the historical development of the functional calculus for linear operators – which contains spectral projections as a special case – we refer to Section 5.2.1 in Pietsch's monograph [Pie07] about the history of Banach spaces and linear operators. A very accessible presentation of spectral projections of matrices and, more generally, of eigenvalue theory via complex analysis techniques can be found, for instance, in [CD13] (an updated version with minor corrections is available on Daniel Daners' webpage).

## Eventual positivity in finite dimensions

### Matrices with eventually positive powers

The predecessors of the eventual positivity theory in finite dimensions can be found in various results about matrices $A$ whose powers $A^k$ are positive for some, or all sufficiently large, $k \in \mathbb{N}$. Matrices with a positive power were, for instance, studied in [Bra61], and more recently in [TCDF15]. Matrices for which a polynomial $p(A)$ is positive are studied in [Sen06]. Some early papers on matrices with eventually positive powers, such as [Fri78, ZT99], were motivated by inverse spectral problems, i.e. the question of which sets in $\mathbb{C}$ can be realised as the spectrum of matrices with certain prescribed properties.

In the 21st century, the literature on eventually positive matrices has grown quickly. Of particular interest were spectral properties of such matrices in the spirit of the Perron–Frobenius theorem, e.g. in the papers [TRH01, JT04, Nou06, ES08, ES09]. In particular, in [Nou06, Theorem 2.2] one can find a discrete-time analogue of the equivalence of (i) and (iv) in Theorem 2.3.1. Matrices with eventually positive powers did not occur in the lecture notes, but they appear in Exercise 2.4. For further references about matrices with eventually positive powers we refer to [Glü16, Section 6.4]; most of the preceding two paragraphs is also taken from this reference.

### Eventually positive matrix semigroups

As for the continuous-time case, eventual positivity of matrix semigroups was studied by Noutsos and Tsatsomeros in [NT08]. The equivalence of (i) and (iv) in Theorem 2.3.1 as well as the characterisation of eventually strongly positive matrix semigroups in Exercise 2.4 appeared in [NT08, Theorem 3.3]; however, the approach outlined in the exercise follows [DGK16, Theorem 6.1]. The fact that eventual positivity implies positivity for $2 \times 2$ matrix semigroups (Exercise 2.5) was observed in [DGK16, Proposition 6.2].

### Perturbation theory

Perturbation theory for matrices with eventually positive powers was studied in [SA17]. In particular, [SA17, Proposition 3.6 and Theorem 3.7] contain discrete-time analogues of the equivalence of (i) and (iii) in Theorem 2.4.2. For the continuous-time case, perturbation theory was first studied in [DG18], though with a focus on the infinite-dimensional case that we consider later. A specific finite-dimensional result in this article is [DG18, Proposition 4.6], which shows that the set of all $A \in \mathbb{R}^{n \times n}$ for which $(e^{tA})_{t \geq 0}$ is eventually strongly positive is open in $\mathbb{R}^{n \times n}$. Theorem 2.4.2 is a finite-dimensional version of [DG18, Theorem 2.3]; we were able to slightly weaken the assumptions in this theorem. Example 2.2.2(b) is also taken from [DG18, Example 2.1]. There, an explicit rank-one operator $B$ is given with the property that $A + sB$ generates an eventually positive semigroup for $s \in [0, 4)$, but the eventual positivity is lost for $s > 4$.

### Eventual positivity with respect to other cones

Positivity of matrices and matrix semigroups has often been studied with respect to general cones. Naturally, this can also be done for eventual positivity; see, for instance, [Kas17, KT17, Soo19]. The following phenomenon in this context is remarkable: Exercise 2.2 suggests that eventual positivity for a dynamical system can be defined 'individually' (i.e. for individual orbits $x \mapsto e^{tA}x$) or 'uniformly' on the level of operators (as in Definition 2.2.1). However, the result of that exercise states that these two notions are equivalent for matrix semigroups. This is a product of two separate features: finite dimensionality, and geometric properties of the positive cone $\mathbb{R}^n_+$. As we see later, the equivalence of individual and uniform eventual positivity fails in infinite dimensions. More surprisingly, it also fails in finite dimensions if we consider positivity with respect to other cones, as was shown in [GH23, Example 3.1].

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