Chapter 13

Characterisations of eventually positive semigroups

13.1 The individual case

The first main result in this chapter characterises individual eventual positivity of semigroups with respect to a quasi-interior point u under sufficiently strong regularity assumptions on the semigroup. It is a semigroup analogue to the characterisation of individually eventually positive resolvents in Theorem 7.3.6.

For a closed linear operator $A\colon X \supseteq \operatorname{dom}(A) \to X$ on a complex Banach space X with $\operatorname{s}(A) \in \mathbb{R}$, we say that $\operatorname{s}(A)$ is a **strictly dominant spectral value of** A if $\operatorname{s}(A) \in \sigma(A)$ and $\operatorname{Re}\lambda < \operatorname{s}(A)$ for all other spectral values λ of A. This resembles a similar notion for eigenvalues that occurred in the finite-dimensional result of Theorem 2.2.3.

Theorem 13.1.1. *Let E be a complex Banach lattice and $u \in E_+$ a quasi-interior point. Let $(e^{tA})_{t\geq 0}$ be a real and eventually norm continuous C_0-semigroup on E and assume that A has compact resolvent and non-empty spectrum. Consider the following conditions:*

(i) *The semigroup $(e^{tA})_{t\geq 0}$ is individually eventually positive with respect to u.*

(ii) *The spectral bound $\operatorname{s}(A)$ is a strictly dominant spectral value of A and its associated spectral projection P satisfies $Pf \succeq u$ whenever $0 \lneq f \in E$.*

(iii) *The spectral bound $\operatorname{s}(A)$ is a strictly dominant spectral value of A, the eigenspace $\ker(\operatorname{s}(A) - A)$ is spanned by a vector $v \succeq u$, and $\ker(\operatorname{s}(A) - A')$ contains a strictly positive functional ψ.*

Each of them implies that $\operatorname{s}(A)$ is an algebraically simple eigenvalue of A and hence a first order pole of $\mathcal{R}(\cdot\,, A)$,[1] *and that $e^{tA} \to P$ in the operator norm as $t \to \infty$.*

One has (i) ⇒ (ii) ⇔ (iii), and if $\operatorname{rg} e^{t_0 A} \subseteq E_u$ for some $t_0 \geq 0$, then all three assertions are equivalent.

[1] By Theorem 6.2.6(b).

Proof. Without loss of generality, we assume $\mathrm{s}(A) = 0$.

"(ii) ⇔ (iii)": This follows from the corresponding result in Theorem 7.3.6.

"(i) ⇒ (iii), *algebraic simplicity, and convergence of* e^{tA}": The argument is along the same lines as the proof of (i)⇒(ii) in Theorem 7.3.6. Assume (i).

Since A has compact resolvent, 0 is a pole of the resolvent (Theorem 6.2.9(a)) of order, say p. Let Q_{-p+1} denote the coefficient of λ^{-p} in the Laurent series expansion of $\mathcal{R}(\lambda, A)$ about 0. We assert that $p = 1$. By Lemma 7.3.3(b), it suffices to show that Q_{-p+1} is positive and $\ker A$ contains a quasi-interior point of E_+.

To this end, recall from the proof of Theorem 11.4.3(a), that the individual eventual positivity implies that Q_{-p+1} is positive. Moreover, by Theorem 11.4.3(a), there exists $0 \lneq v \in \ker A$. The individual eventual strong positivity with respect to the quasi-interior point u implies that there exists $t > 0$ such that $v = e^{tA} v \succeq u$. In particular, v is also a quasi-interior point of E_+.

Hence, the characterisation for convergence of eventually positive semigroups (Theorem 12.4.1) ensures that $e^{tA} \to P$ in the operator norm as $t \to \infty$. In particular, P is positive. The proof of Theorem 12.4.1 or alternatively, an application of Theorem 12.3.2 also shows that 0 is a strictly dominant spectral value.

From Theorem 11.4.3(a), there exists $0 \lneq \psi \in \ker A'$. It remains to show that ψ is strictly positive. For this, let $0 \lneq x \in E$ and choose $\tau > 0$ such that $e^{\tau A} x \succeq u$. Then

$$\langle \psi, x \rangle = \left\langle e^{\tau A'} \psi, x \right\rangle = \langle \psi, e^{\tau A} x \rangle \geq \langle \psi, u \rangle > 0$$

because u is a quasi-interior point (Proposition 7.1.4).

Lastly, assume that $\operatorname{rg} e^{t_0 A} \subseteq E_u$ for some $t_0 \geq 0$.

"(ii) ⇒ (i)": Since A has compact resolvent, each spectral value is a pole of $\mathcal{R}(\cdot, A)$ by Theorem 6.2.9(a). Moreover, the positivity property of P ensures that 0 is a first order pole (Theorem 7.3.6). We can thus conclude from the characterisation of convergence of semigroups (Theorem 12.3.2) that $e^{tA} \to P$ in the operator norm as $t \to \infty$.

On the other hand, as 0 is a first order pole, Theorem 6.2.6(b) and (c) show that $\operatorname{rg} P = \ker A$, so $A|_{\operatorname{rg} P}$ is the zero operator on $\operatorname{rg} P$. Hence, $e^{tA} P = P$ for all $t \geq 0$. Also, by the closed graph theorem, $e^{t_0 A} \in \mathcal{L}(E, E_u)$. Wherefore for each $0 \lneq x \in E$,

$$e^{tA} x = e^{t_0 A} e^{(t-t_0)A} x \to e^{t_0 A} P x = P x \quad \text{as } t \to \infty$$

in E_u. The assertion is now a consequence of Lemma 7.3.5. □

Example 11.1.2 demonstrated that there are C_0-semigroups that are individually but not uniformly eventually positive. It is natural to ask whether additional compactness assumptions, for instance on the semigroup operators, makes individual and uniform eventual positivity equivalent. The answer is negative, as the following alienated version of Example 11.1.2 shows. It also serves nicely as an application of Theorem 13.1.1.

Example 13.1.2. There exists a real C_0-semigroup $(e^{tA})_{t\geq 0}$ on a Banach lattice E with the following properties:

(a) The generator A has compact resolvent. Moreover, for each $t > 0$ the operator e^{tA} is compact and satisfies $\operatorname{rg} e^{tA} \subseteq \operatorname{dom}(A)$.

(b) $(e^{tA})_{t\geq 0}$ is individually eventually positive with respect to $\mathbb{1}$.

(c) $(e^{tA})_{t\geq 0}$ is not uniformly eventually positive with respect to 0.

Proof. Let $\mathrm{c} := \mathrm{c}(\mathbb{Z})$ be the Banach lattice of all complex-valued convergent sequences on $\mathbb{Z}$ endowed with the supremum norm and let $\mathrm{c}_0 \subseteq \mathrm{c}$ consist of those sequences that converge to 0. Since each sequence $(x_n) \in \mathrm{c}_0$ can be written as

$$(x_n) = \frac{(x_n) + (x_{-n})}{2} + \frac{(x_n) - (x_{-n})}{2},$$

we have the decomposition $\mathrm{c} = \mathbb{C}\,\mathbb{1} \oplus \mathrm{c}_0 = \mathbb{C}\,\mathbb{1} \oplus \mathrm{c}_0^s \oplus \mathrm{c}_0^a$, where

$$\mathrm{c}_0^s := \{(x_n) \in \mathrm{c}_0 : x_{-n} = x_n \ \forall\, n \in \mathbb{Z}\} \quad \text{and} \quad \mathrm{c}_0^a := \{(x_n) \in \mathrm{c}_0 : x_{-n} = -x_n \ \forall\, n \in \mathbb{Z}\}$$

are the subspaces of all symmetric and anti-symmetric sequences in c_0.

On c, consider the block diagonal operator

$$D := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -M_\beta & 0 \\ 0 & 0 & -M_\alpha \end{pmatrix} \in \mathcal{L}(\mathbb{C}\,\mathbb{1} \oplus \mathrm{c}_0^s \oplus \mathrm{c}_0^a);$$

where $\alpha := (\alpha_n), \beta := (\beta_n)$ are chosen to be strictly positive symmetric sequences such that $\alpha_n, \beta_n \to \infty$ as $|n| \to \infty$ and $\alpha_n < \beta_n$ for all $n \in \mathbb{N}$. This in particular implies that

$$e^{-n\beta_n} - e^{-n\alpha_n} < 0 \qquad \forall\, n \in \mathbb{N}. \tag{13.1.1}$$

In addition, consider the block diagonal operator on $\mathrm{c} = \mathbb{C}\,\mathbb{1} \oplus \mathrm{c}_0$ given by

$$B \begin{pmatrix} b & 0 \\ 0 & x \end{pmatrix} := \begin{pmatrix} b + \langle g, x \rangle & 0 \\ 0 & x \end{pmatrix} \qquad \forall\, b\,\mathbb{1} + x \in \mathbb{C}\,\mathbb{1} + \mathrm{c}_0;$$

where $g := (g_n) \in \ell^1(\mathbb{Z}) \cap \mathrm{c}_0^s$ is chosen such that $g_n > 0$ for all $n \in \mathbb{Z}$, $\|g\|_{\ell^1} = 1$, and

$$2g_n + e^{-n\beta_n} - e^{-n\alpha_n} < 0 \tag{13.1.2}$$

for sufficiently large $n \in \mathbb{N}$.

It is shown in Exercise 13.4(b) and (f) that $-M_\alpha$ and $-M_\beta$ are semigroup generators, so D generates a C_0-semigroup $(e^{tD})_{t\geq 0}$ on c. Hence, the operator $A := B^{-1}DB$ generates the semigroup $(B^{-1}e^{tD}B)_{t\geq 0}$ on c (see Exercise 13.3 for similarity transforms of C_0-semigroups). Let us now show that A has the claimed properties (a)– (c):

(a) For the semigroups generated by the multiplication operators M_α and M_β those properties follow from Exercise 13.4(e). All three properties are preserved by taking direct sums and by similarity transforms, so they are true for $(e^{tA})_{t\geq 0}$ as well.

(b) It follows from (a) and from Proposition 12.2.3 (or Proposition 12.2.6) that $(e^{tA})_{t\geq 0}$ is eventually norm continuous. Moreover, $\sigma(A)$ is non-empty since $0 \in \sigma(A)$. Also, $\operatorname{rg} e^{tA} \subseteq c = c_{\mathbb{1}}$ for all $t > 0$. Thus, the conditions (i)–(iii) in Theorem 13.1.1 are equivalent for the semigroup $(e^{tA})_{t\geq 0}$ and $u = \mathbb{1}$. We show that condition (iii) is satisfied.

Exercise 13.4(a) implies that $\sigma(M_\alpha) \subseteq (-\infty, 0)$ and $\sigma(M_\beta) \subseteq (-\infty, 0)$, so 0 is a strictly dominant spectral value of A. A brief computation shows that the associated spectral projection P is given by

$$P(b\mathbb{1} + x) = \big(b + \langle g, x\rangle\big)\mathbb{1} = \langle g, b\mathbb{1} + x\rangle \mathbb{1}$$

for every $b \in \mathbb{C}$ and $x \in c_0$; here we have used that $\langle g, \mathbb{1}\rangle = \|g\|_{\ell^1} = 1$. In other words, $P = \mathbb{1} \otimes g$. So indeed, $Py \succeq \mathbb{1}$ for each $0 \lneq y \in c$ since each entry of g is strictly positive.

(c) Let $n \in \mathbb{N}$ be such that (13.1.2) holds and let $e^{(n)} \in c_0$ be the vector with 1 in the n^{th} position and 0 elsewhere. We compute

$$\begin{aligned} 2e^{nA}e^{(n)} &= \begin{pmatrix} \langle g, 2e^{(n)} - e^{-nM_\beta}\left(e^{(n)} + e^{(-n)}\right)\rangle & 0 & 0 \\ 0 & e^{-nM_\beta}\left(e^{(n)} + e^{(-n)}\right) & 0 \\ 0 & 0 & e^{-nM_\alpha}\left(e^{(n)} - e^{(-n)}\right) \end{pmatrix} \\ &\leq \begin{pmatrix} 2\langle g, e^{(n)}\rangle & 0 & 0 \\ 0 & e^{-nM_\beta}\left(e^{(n)} + e^{(-n)}\right) & 0 \\ 0 & 0 & e^{-nM_\alpha}\left(e^{(n)} - e^{(-n)}\right) \end{pmatrix}. \end{aligned}$$

In particular,

$$\left(e^{nA}e^{(n)}\right)_{-n} \leq \frac{1}{2}\left(2g_n + e^{-n\beta_n} - e^{-n\alpha_n}\right) < 0$$

and so $(e^{tA})_{t\geq 0}$ is not uniformly eventually positive with respect to 0. □

13.2 The uniform case

Similarly as for resolvents (Corollary 9.1.3), one gets a characterisation of uniform eventual positivity if one adds a smoothing assumption on the dual semigroup.

Theorem 13.2.1. *Let E be a complex Banach lattice, $u \in E_+$ a quasi-interior point, and $\varphi \in E'$ be a strictly positive functional. Let $(e^{tA})_{t\geq 0}$ be a real and eventually norm continuous C_0-semigroup and assume that A has compact resolvent and non-empty spectrum.*

If $\operatorname{rg} e^{t_1 A} \subseteq E_u$ and $\operatorname{rg} e^{t_2 A'} \subseteq E'_\varphi$ for some $t_1, t_2 \geq 0$, then the following are equivalent:

(i) *The semigroup $(e^{tA})_{t\geq 0}$ is uniformly eventually positive with respect to $u \otimes \varphi$.*

(ii) *The spectral bound $\mathrm{s}(A)$ is a strictly dominant spectral value of A and its associated spectral projection P satisfies $P \geq u \otimes \varphi$.*

(iii) *The spectral bound* $\operatorname{s}(A)$ *is a strictly dominant spectral value of* A*, the eigenspace* $\ker(\operatorname{s}(A) - A)$ *is spanned by a vector* $v \succeq u$*, and* $\ker(\operatorname{s}(A) - A')$ *contains a strictly positive functional* $\psi \succeq \varphi$.

Proof. Without loss of generality, we assume $\operatorname{s}(A) = 0$.

"(ii) ⇔ (iii)": This can be easily deduced from Theorem 7.3.6.

"(i) ⇒ (iii)": By Theorem 13.1.1, 0 is a strictly dominant spectral value, $\ker A$ is spanned by a vector $v \succeq u$, and there exists a strictly positive $\psi \in \ker A'$. Choosing $t > 0$ sufficiently large, we get $\psi = e^{tA'}\psi = (e^{tA})'\psi \succeq (u \otimes \varphi)'\psi = \langle \psi, u \rangle \varphi \succeq \varphi$.

"(ii) ⇒ (i)": By Theorem 13.1.1, 0 is an algebraically simple eigenvalue, a first order pole, and $e^{tA} \to P$ in the operator norm as $t \to \infty$. In particular, $e^{tA}P = P$ for each $t \geq 0$.

Further, the assumptions $\operatorname{rg} e^{t_1 A} \subseteq E_u$ and $\operatorname{rg} e^{t_2 A'} \subseteq E'_\varphi$ imply by Corollary 8.2.4 that

$$e^{(t+t_1+t_2)A} - P = e^{t_1 A}(e^{tA} - P)e^{t_2 A} \in \mathcal{L}(E^\varphi, E_u)$$

for each $t > 0$. It follows that $e^{tA} \to P$ in $\mathcal{L}(E^\varphi, E_u)$ as $t \to \infty$. Since $P \succeq u \otimes \varphi$, one can deduce that $e^{tA} \succeq u \otimes \varphi$ for sufficiently large t. □

For operators associated to symmetric sesquilinear forms on L^2 – in other words, for self-adjoint operators – Theorem 13.2.1 takes quite a simple form.

Corollary 13.2.2. *Let* $H = L^2(\Omega, \nu)$ *for a* σ*-finite measure space* (Ω, ν) *and let* V *be a complex Hilbert space such that* V *embeds continuously, densely, and compactly into* H*. Let* $\mathfrak{a}\colon V \times V \to \mathbb{C}$ *be a bounded, real, and symmetric sesquilinear form on* V *that satisfies the ellipticity estimate*

$$\operatorname{Re} \mathfrak{a}(v, v) + \mu \|v\|_H^2 \geq \delta \|v\|_V^2$$

for some $\mu \in \mathbb{R}, \delta > 0$ *and for all* $v \in V$.

Let $u \in H_+$ *such that the operator* $A\colon H \supseteq \operatorname{dom}(A) \to H$ *associated to the form* $\mathfrak{a}$ *satisfies* $\operatorname{dom}(A^m) \subseteq H_u$ *for some* $m \in \mathbb{N}_0$. [2]

(i) $(e^{tA})_{t\geq 0}$ *is individually eventually positive with respect to* u.

(ii) $(e^{tA})_{t\geq 0}$ *is uniformly eventually positive with respect to* $u \otimes u$.

(iii) *The spectral projection* P *associated to* $\operatorname{s}(A)$ *satisfies* $Pf \succeq u$ *whenever* $0 \lneq f \in E$.

(iv) *The eigenspace* $\ker(\operatorname{s}(A) - A)$ *is spanned by a vector* $v \succeq u$.

The equivalence of individual and uniform eventual positivity above should not come as as surprise because symmetric forms correspond to self-adjoint generators and we've seen in Theorem 13.2.1 that to go from individual to uniform eventual positivity, it suffices that the dual operator behaves nicely as well.

[2] In particular, u is a quasi-interior point of H_+ since A^m is densely defined (Exercise 9.1).

Proof of Corollary 13.2.2. First of all, since the form $\mathfrak{a}$ is symmetric, $\sigma(A)$ is non-empty (Proposition 6.2.10(c)) and contained in $(-\infty, \mathrm{s}(A))$ (Theorem 5.1.4(c)). As $\sigma(A)$ is closed, it follows that $\mathrm{s}(A)$ is a strictly dominant spectral value. Additionally, compactness of the embedding $V \hookrightarrow H$ ensures that A has compact resolvent due to Proposition 6.2.10(b). Consequently, $\mathrm{s}(A)$ is even an eigenvalue (Theorem 6.2.9(a)).

Next, since the form $\mathfrak{a}$ is symmetric, the operator A is real (Proposition 5.1.6) and self-adjoint (Proposition 8.4.3). In particular, the semigroup $(e^{tA})_{t\geq 0}$ is also real. Moreover, the assumption $\operatorname{rg} e^{tA} \subseteq \operatorname{dom}(A^m) \subseteq H_u$ holds for all $t > 0$ due to Corollary 12.1.5. In turn, $(e^{tA})_{t\geq 0}$ is eventually norm continuous by Proposition 12.2.3. [3]

The equivalences thus follow by employing Theorems 13.1.1 and 13.2.1. □

We illustrate the characterisation of uniform eventual positivity by different differential operators in the following two examples.

Example 13.2.3 (The Laplacian with non-local boundary conditions, re-revisited)**.** Consider our friend the Laplace operator $\Delta_B \colon L^2(0,1) \supseteq \operatorname{dom}(\Delta_B) \to L^2(0,1)$ with non-local boundary conditions, whose domain is

$$\operatorname{dom}(\Delta_B) = \left\{ u \in H^2(0,1) : \begin{pmatrix} -u'(0) \\ u'(1) \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u(0) \\ u(1) \end{pmatrix} \right\}.$$

We know that Δ_B is the operator associated to the symmetric form $\mathfrak{a}$ in Exercise 5.6 for the choice $B := -\frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. By Examples 10.2.6 and 10.3.6, Δ_B generates a non-positive C_0-semigroup. Nevertheless, $(e^{t\Delta_B})_{t\geq 0}$ is uniformly eventually positive with respect to $\mathbb{1} \otimes \mathbb{1}$.

Proof. All the assumptions of Corollary 13.2.2 hold: We showed in Example 10.2.6 that Δ_B is associated to a symmetric form satisfying the assumptions of Theorem 5.1.4. The form domain is $H^1(0,1)$, that embeds compactly into $L^2(0,1)$ by Theorem 6.3.1. Clearly $\mathbb{1}$ is a quasi-interior point of $L^2(0,1)_+$. We know that $\operatorname{dom}(\Delta_B) \subseteq H^2(0,1) \hookrightarrow L^\infty(0,1) = L^2(0,1)_{\mathbb{1}}$; indeed, the 1-dimensional Sobolev embedding theorem (Theorem 5.3.7) even gives $H^1(0,1) \hookrightarrow L^\infty(0,1)$.

Finally, recall from Example 8.3.5 that $\mathrm{s}(\Delta_B) < 0$ is a strictly dominant spectral value and the corresponding eigenspace is spanned by a vector $v \succeq \mathbb{1}$. The asserted uniform eventual positivity is thus a consequence of Corollary 13.2.2. □

Example 13.2.4 (Minus the square of the Dirichlet Laplacian)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be a bounded, open, and connected set. For the sake of simplicity, assume that Ω has C^∞-boundary, and consider once more the Dirichlet Laplacian $\Delta_{\mathrm{Dir}} \colon \operatorname{dom}(\Delta_{\mathrm{Dir}}) \subseteq L^2(\Omega) \to L^2(\Omega)$. In Example 7.3.8, we showed that $\mathrm{s}(\Delta_{\mathrm{Dir}}) < 0$ is a strictly dominant eigenvalue and the corresponding eigenspace is spanned by a function $v \succeq u := \operatorname{dist}(\cdot, \partial\Omega)$.

The semigroup $(e^{-t\Delta_{\mathrm{Dir}}^2})_{t\geq 0}$ is uniformly eventually positive with respect to $u \otimes u$ but not positive.

[3] In fact, a glance at the proof of Proposition 12.2.3 shows that the semigroup is even norm continuous on $(0,\infty)$.

Proof. The non-positivity of the semigroup is outsourced to Exercise 13.2.

By Exercise 9.4(a), $-\Delta_{\text{Dir}}^2$ is associated to the form $\mathfrak{a}\colon \operatorname{dom}(\Delta_{\text{Dir}}) \times \operatorname{dom}(\Delta_{\text{Dir}}) \to \mathbb{C}$ given by

$$\mathfrak{a}(v,w) := (\Delta_{\text{Dir}} v \mid \Delta_{\text{Dir}} w)_{L^2(\Omega)} \qquad \forall\, v,w \in \operatorname{dom}(\Delta_{\text{Dir}}). \tag{13.2.1}$$

Since Δ_{Dir} is self-adjoint, the form $\mathfrak{a}$ is symmetric. Observe that

$$\operatorname{Re}\mathfrak{a}(v,v) + \|v\|_{L^2(\Omega)}^2 = \|\Delta_{\text{Dir}} v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 = \|v\|_{\operatorname{dom}(\Delta_{\text{Dir}})}^2 \qquad \forall\, v \in \operatorname{dom}(\Delta_{\text{Dir}});$$

so $\mathfrak{a}$ satisfies the ellipticity estimate in Corollary 13.2.2. The form domain is $\operatorname{dom}(\Delta_{\text{Dir}})$, which we already know embeds compactly into $L^2(\Omega)$ (since the Dirichlet Laplacian has compact resolvent, see Example 6.3.5). Hence the compactness of the resolvent of $-\Delta_{\text{Dir}}^2$ follows from Proposition 6.2.10(b). The smoothness of the boundary now enters the picture as it allows us to apply Example 7.2.4(a) to deduce $\operatorname{dom}\big(\Delta_{\text{Dir}}^m\big) \subseteq L^2(\Omega)_u$ for some $m \in \mathbb{N}$. Thus we have verified all the assumptions of Corollary 13.2.2.

It remains to show that $(e^{-t\Delta_{\text{Dir}}^2})_{t\ge 0}$ is individually eventually positive with respect to u. We show that condition (iv) in Corollary 13.2.2 holds. In fact, we verify that $\operatorname{s}(-\Delta_{\text{Dir}}^2) = -\operatorname{s}(\Delta_{\text{Dir}})^2$ and that the corresponding eigenspace is spanned by v as well.

Set $\lambda_0 := \operatorname{s}(\Delta_{\text{Dir}})$. Then clearly $-\Delta_{\text{Dir}}^2 v = -\lambda_0^2 v$, so v is an eigenfunction of $-\Delta_{\text{Dir}}^2$ corresponding to the eigenvalue $-\lambda_0^2$. On the other hand, if $w \in \operatorname{dom}(\Delta_{\text{Dir}})$ is such that $-\Delta_{\text{Dir}}^2 w = -\lambda_0^2 w$, then

$$0 = (\lambda_0 + \Delta_{\text{Dir}})(\lambda_0 - \Delta_{\text{Dir}})\, w.$$

As the operator $\lambda_0 + \Delta_{\text{Dir}} = -(-\lambda_0 - \Delta_{\text{Dir}})$ is invertible (since $-\lambda_0 > 0$), it follows that $w \in \ker(\lambda_0 - \Delta_{\text{Dir}})$. But then w is a scalar multiple of v, since we have proved in Example 7.3.8(a) that $\ker(\lambda_0 - \Delta_{\text{Dir}})$ is spanned by v. □

Exercises for Chapter 13

Exercise 13.1. Consider the C_0-semigroup $(e^{tA})_{t\ge 0}$ from Example 11.1.2, but this time on the space $L^p(-1,1)$ for $p \in [1,\infty)$.

(a) Show that $(e^{tA})_{t\ge 0}$ is not individually eventually positive with respect to 0.

(b) Show that 0 is an isolated and strictly dominant spectral value of A and that the associated spectral projection P satisfies $Pf \succeq \mathbb{1}$ for all $0 \lneq f \in L^p(-1,1)$.

(c) Show that A does not have compact resolvent and that

$$\operatorname{rg} e^{tA} \not\subseteq L^\infty(-1,1) = L^p(-1,1)_{\mathbb{1}}$$

for every $t \ge 0$.[4]

Exercise 13.2. Under the assumptions of Example 13.2.4, show that $(e^{-t\Delta_{\mathrm{Dir}}^2})_{t\ge 0}$ is not a positive semigroup.

Hint: Use the Beurling–Deny criterion in Corollary 10.3.4 and the regularity result for $\operatorname{dom}(\Delta_{\mathrm{Dir}})$ in Theorem 5.3.2(a).

Exercise 13.3. Let X and Y be Banach space over the same field, let $\Phi \in \mathcal{L}(X,Y)$ be bijective, and let $(e^{tA})_{t\ge 0}$ be a C_0-semigroup on X. Show that

$$\big(\Phi e^{tA}\Phi^{-1}\big)_{t\ge 0}$$

is a C_0-semigroup on Y whose generator $B\colon Y \supseteq \operatorname{dom}(B) \to Y$ is given by

$$\operatorname{dom}(B) = \Phi\big(\operatorname{dom}(A)\big), \qquad By = \Phi A \Phi^{-1} y.$$

Exercise 13.4. Consider the space $c_0 := c_0(\mathbb{Z})$ of complex-valued sequences $(x_n)_{n\in\mathbb{Z}}$ that satisfy $x_n \to 0$ as $|n| \to \infty$. Let $\alpha = (\alpha_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers. We define a linear operator $M_\alpha\colon\ c_0 \supseteq \operatorname{dom}(M_\alpha) \to c_0$ by

$$\operatorname{dom}(M_\alpha) = \{x \in c_0 : \alpha x \in c_0\}, \qquad M_\alpha x = \alpha x,$$

where products of sequences are defined componentwise.

[4]Since A does not have compact resolvent, Theorem 13.1.1 cannot be applied to A – however this is actually an artifact of the techniques that we used to prove this theorem. By more advanced techniques one can show that Theorem 13.1.1 remains true if A does not have compact resolvents, but $\sigma(A) \cap \big(\mathrm{s}(A) + i\mathbb{R}\big)$ consists of pole of the resolvent only. So the actual reason why the implication (ii) ⇒ (ii) from Theorem 13.1.1 fails in this example is that the semigroup operators do not map into the principle ideal $L^p(-1,1)_{\mathbb{1}}$.

(a) Determine the spectrum and the eigenvalues of M_α.

Hint: Compare Exercise 3.2(c).

(b) Show that M_α generates a C_0-semigroup on c_0 if and only if $\sup_{n\in\mathbb{Z}} \operatorname{Re} \alpha_n < \infty$. In this case, compute the operator e^{tM_α} for each $t \geq 0$.

(c) Assume that M_α generates a C_0-semigroup. Show that $\operatorname{rg} e^{tM_\alpha} \subseteq \operatorname{dom}(M_\alpha)$ for every $t > 0$. Is $(e^{tM_\alpha})_{t\geq 0}$ eventually norm continuous?

(d) Show that a set $S \subseteq c_0$ is relatively compact if and only if the exists $y \in c_0$ such that $|x| \leq y$ for all $x \in S$.

(e) Assume that M_α generates a C_0-semigroup and let $t_0 > 0$. Show that the following are equivalent:

  (i) e^{tM_α} is a compact operator on c_0 for each $t > 0$.

  (ii) $e^{t_0 M_\alpha}$ is a compact operator on c_0.

  (iii) $\operatorname{Re} a_n \to -\infty$ as $|n| \to \infty$.

  (iv) M_a has compact resolvent.

(f) Assume that M_α generates a C_0-semigroup and that α is symmetric, i.e. $\alpha_n = \alpha_{-n}$ for all $n \in \mathbb{Z}$. Let

$$c_0^s := \{(x_n) \in c_0 : x_{-n} = x_n \ \forall\ n \in \mathbb{Z}\} \quad \text{and} \quad c_0^a := \{(x_n) \in c_0 : x_{-n} = -x_n \ \forall\ n \in \mathbb{Z}\}$$

Show that e^{tM_a} leaves c_0^s and c_0^a invariant and determine the generators of the restrictions of $(e^{tM_a})_{t\geq 0}$ to those two spaces.

Notes for Chapter 13

The theory of eventually positive semigroups was developed in the previous decade with contributions by Daners, Kennedy, the ISEM 29 lecturers, and many others. An overview of the topic as of 2022 can be found in [Glü22]. To keep the self-praise at an acceptable level, let us try to be brief in the following notes.

A theory for eventually positive semigroups

Before the development of the general theory, eventual positivity (and local versions thereof) was observed for the semigroup generated by $-\Delta^2$ on the whole space $\mathbb{R}^n$ in [FGG08, GG08]. For the semigroup generated by the Dirichlet-to-Neumann operators on a disk, Daners showed in [Dan14] that eventual positivity occurs for certain parameters choices.

The general theory for eventually positive semigroups began where the ISEM ended with this chapter. The characterisation of individual eventual positivity in Theorem 13.1.1 is given in in the special case of $C(K)$-spaces in [DGK16b]. On general Banach lattices it was proved in [DGK16a], with the caveat that the smoothing assumption $e^{tA}E \subseteq E_u$ was also needed there for the implication (i) ⇒ (ii). The smoothing assumption was then shown in [DG17] to be unnecessary for this implication. Example 13.1.2 is also taken from [DGK16a]. A characterisation of uniform eventual positivity for self-adjoint semigroups on L^2 (Corollary 13.2.2) was given in [Glü16, Theorem 10.2.1]. This was generalised to non-self-adjoint semigroups and to general Banach lattices (Theorem 13.2.1) in [DG18a]. The non-local Robin boundary conditions from Example 13.2.3 can be treated in much more generality [GM24].

Further topics about eventual positivity

To keep the amount of material at a reasonable level, we only touched upon a subset of topics related to eventual positivity. Topics not treated in the ISEM include perturbation theory in infinite dimensions [AM25, DG18b, PRS25], local eventual positivity (see the articles [Aro22, DGM23, Mui23b] and the PhD theses [Aro23, Mui23a]), irreducibility of eventually positive semigroups [AG24, Aro25b], and eventual domination of semigroups [GM21, AG23b]. Results about the growth behaviour of eventually positive semigroups that are more subtle than the ones treated in Section 11.2 can be found in [AC23].

Applications

Throughout the lecture notes we presented a small selection of differential operators that generate eventually positive semigroups or whose resolvent is eventually positive. The choice which operators to include was mainly made based on whether we could provide or at least explain the necessary tools for their analysis within the ISEM lecture notes.

There is a large variety of further examples for which eventual positivity has been shown in the literature. They include various delay differential equations (see [DGK16b, Section 6.5], [DGK16a, Section 9], [Glü16, Sections 8.4 and 11.6], and [PRS25]), various differential operator with Wentzell boundary conditions [DKP21, KMP25, Plo24], fourth order differential operators with unbounded coefficients [AGRT22], and differential operators on metric graphs (see e.g. [GM20a, GM20b]).

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