Chapter 12

Convergence to equilibrium

12.1 Smoothing by C_0-semigroups

Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a Banach space X. In this section, we discuss conditions on the operators e^{tA} to map X into $\operatorname{dom}(A)$ for $t>0$. We are interested in this property for three reasons. Firstly, we will give characterisations of eventual positivity in the next chapter that are similar to those given for the resolvent in Chapters 7 and 8. For the resolvent, that characterisation is closely tied to the property that the range of the resolvent is contained in $\operatorname{dom}(A)$, which indicates that a similar property will be useful for semigroups operators. Secondly, the property $\operatorname{rg} e^{tA} \subseteq \operatorname{dom}(A)$ is related to differentiability of mild solutions by the following proposition.

Proposition 12.1.1. *Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a Banach space X, let $x \in X$, and lwt $t_0 \geq 0$. The following assertions are equivalent.*

(i) *The orbit map $[0,\infty) \to X$, $t \mapsto e^{tA}x$ is differentiable on $[t_0,\infty)$.*

(ii) *$e^{t_0 A}x \in \operatorname{dom}(A)$.*

If (i)–(ii) are satisfied, then $e^{tA}x \in \operatorname{dom}(A)$ and $\frac{\mathrm{d}}{\mathrm{d}t}e^{tA}x = Ae^{tA}x$ hold for all $t \in [t_0,\infty)$.

Proof. "(i) ⇒ (ii)": The map $s \mapsto e^{(s+t_0)A}x = e^{sA}e^{t_0A}x$ is differentiable on $[0,\infty)$, and in particular at $s=0$. The definition of the domain of a semigroup generator (Definition 10.2.1(b)) thus implies $e^{t_0A}x \in \operatorname{dom}(A)$ and $\frac{\mathrm{d}}{\mathrm{d}t}e^{tA}x|_{t=t_0} = Ae^{t_0A}x$.

The same argument works if t_0 is replaced with any larger real number. This shows the property claimed at the end of the proposition.

"(ii) ⇒ (i)": Assume that $e^{t_0A}x \in \operatorname{dom}(A)$. The last part of Theorem 10.1.7 then shows that the mapping $[0,\infty) \to X$, $s \mapsto e^{(s+t_0)A}x = e^{sA}e^{t_0A}x$ is differentiable with the derivative $Ae^{sA}e^{t_0A}x = Ae^{(s+t_0)A}x$ at each $s \geq 0$. □

A third reason why $e^{tA}X \subseteq \operatorname{dom}(A)$ is that it implies another useful property, the so-called **eventual norm continuity** of a C_0-semigroup that is discussed in Section 12.2.

In the rest of this section, we derive conditions for the property $\operatorname{rg} e^{tA} \subseteq \operatorname{dom}(A)$ and, more generally, $\operatorname{rg} e^{tA} \subseteq \operatorname{dom}\left(A^k\right)$, to hold for all $k \in \mathbb{N}$ and $t > 0$. We need the following Fourier-like representation formula for the semigroup in terms of the resolvent.

Proposition 12.1.2. *Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a complex Banach space X with $\omega_0(A) < 0$ and let $2 \leq k \in \mathbb{N}$. If $x \in X$ is such that $\left\|\beta^k \mathcal{R}(\mathrm{i}\beta, A)^k x\right\|$ is bounded for all sufficiently large $|\beta|$, then*

$$t^{k-1} e^{tA} x = \frac{(k-1)!}{2\pi} \int_{\mathbb{R}} e^{\mathrm{i}t\beta} \mathcal{R}(i\beta, A)^k x \,\mathrm{d}\beta \quad \forall\ t \geq 0.$$

Before we prove the proposition, let us show in the following remark that the norm estimate for $\left\|\mathcal{R}(\mathrm{i}\beta, A)^k x\right\|$ in the theorem is automatically true if $x \in \operatorname{dom}\left(A^k\right)$. We do not need this fact in the course, but we consider it worthwhile to include it for context.

Remark 12.1.3. Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a complex Banach space X with $\omega_0(A) < 0$ and let $k \in \mathbb{N}_0$. If $x \in \operatorname{dom}\left(A^k\right)$, then $\left\|\beta^k \mathcal{R}(\mathrm{i}\beta, A)^k x\right\|$ is for bounded for all sufficiently large $|\beta|$.

Proof. First, define $C := \sup_{\beta\in\mathbb{R}} \left\|\mathcal{R}(\mathrm{i}\beta, A)\right\|$; the assumption $\omega_0(A) < 0$ implies that $C < \infty$, since in Theorem 10.2.3(iv) one can take any $\omega > \omega_0(A)$. Therefore, if $|\beta| > 1$, then using the resolvent identity

$$\begin{aligned}\left\|\beta^k \mathcal{R}(\mathrm{i}\beta, A)^k x\right\| &= \left\|\beta^k \mathcal{R}(\mathrm{i}\beta, A)^k \left(\mathcal{R}(0, A)^k (-A)^k x\right)\right\| \leq \left\|\beta^k \left(\mathcal{R}(\mathrm{i}\beta, A)\mathcal{R}(0, A)\right)^k\right\| \left\|A^k x\right\| \\ &= \left\|\left(\mathcal{R}(\mathrm{i}\beta, A) - \mathcal{R}(0, A)\right)^k\right\| \left\|A^k x\right\| \leq \left(C + \|\mathcal{R}(0, A)\|\right)^k \left\|A^k x\right\|. \end{aligned}$$ □

Proof of Proposition 12.1.2. Note that the integrand is Bochner integrable since $k \geq 2$.

To show the equality for $t^{k-1} e^{tA} x$, we use some standard results for the scalar-valued Fourier transform on $\mathbb{R}$. Let $\mathcal{F}\colon L^1(\mathbb{R}) \to C_0(\mathbb{R})$ be given by

$$(\mathcal{F}f)(\beta) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\mathrm{i}t\beta} f(t) \,\mathrm{d}t$$

for all $f \in L^1(\mathbb{R})$ and $\beta \in \mathbb{R}$. It has, among others, the following two properties:

(a) If $f \in L^1(\mathbb{R})$ and $\mathcal{F}f \in L^1(\mathbb{R})$, then $(\mathcal{F}^2 f)(t) = f(-t)$ for all $t \in \mathbb{R}$.

(b) If $f \in L^1(\mathbb{R})$ and the mappings $g_\ell\colon \mathbb{R} \to \mathbb{C}$, $t \mapsto t^\ell f(t)$ are also in $L^1(\mathbb{R})$ for all $\ell \in \mathbb{N}$, then $\mathcal{F}f \in C^\infty(\mathbb{R})$ and $\mathcal{F}g_\ell = \mathrm{i}^\ell (\mathcal{F}f)^{(\ell)}$ for all $\ell \in \mathbb{N}$.

Now fix a functional $x' \in X'$ and define $f\colon \mathbb{R} \to \mathbb{C}$ by

$$f(t) := \begin{cases} \langle x', e^{tA} x\rangle & \text{if } t \geq 0, \\ 0 & \text{if } t \in (-\infty, 0). \end{cases}$$

Moreover, let $g_\ell\colon \mathbb{R} \to \mathbb{C}$, $g_\ell(t) = t^\ell f(t)$ for each $\ell \in \mathbb{N}$, as in property (b) above. Since $\omega_0(A) < 0$, the functions f and g_ℓ are in $L^1(\mathbb{R};\mathbb{C})$ for each $\ell \in \mathbb{N}$.

It follows also from $\omega_0(A) < 0$ and from the Laplace transform representation of $\mathcal{R}(\cdot, A)$ (Theorem 10.2.3(b)) that $(\mathcal{F}f)(\beta) = \frac{1}{\sqrt{2\pi}} \langle x', \mathcal{R}(\mathrm{i}\beta, A)x\rangle$ and thus,

$$(\mathcal{F}g_{k-1})(\beta) = \mathrm{i}^{k-1}(\mathcal{F}f)^{(k-1)}(\beta) = \frac{(k-1)!}{\sqrt{2\pi}} \left\langle x', \mathcal{R}(\mathrm{i}\beta, A)^k x\right\rangle$$

for all $\beta \in \mathbb{R}$; the first equality follows from property (b) above and the second equality is due to the identify $\mathcal{R}^{(k-1)}(\lambda, A) = (-1)^{k-1}(k-1)!\mathcal{R}(\lambda, A)^k$ (which one can, for instance, obtain from the Taylor serious expansion in Proposition 3.3.2(a)). Finally, we apply property (a) above to the function g_{k-1} and evaluate to get

$$\begin{aligned} g_{k-1}(t) = (\mathcal{F}^2 g_{k-1})(-t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mathrm{i}t\beta} (\mathcal{F}g_{k-1})(\beta)\,\mathrm{d}\beta \\ &= \frac{(k-1)!}{2\pi} \int_{\mathbb{R}} e^{\mathrm{i}t\beta} \langle x', \mathcal{R}(\mathrm{i}\beta, A)x\rangle^k \,\mathrm{d}\beta \end{aligned}$$

for all $t \geq 0$. Since $x' \in X'$ was arbitrary, this proves the claim. □

As a consequence of the previous proposition, one easily gets the following condition for e^{tA} to map into the domain of every power of A.

Theorem 12.1.4. *Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a complex Banach space X. Assume that there exists $\omega > \omega_0(A)$ such that $\|\beta\mathcal{R}(\omega + \mathrm{i}\beta, A)\|$ is bounded for all $\beta \in \mathbb{R}$ with sufficiently large modulus. Then $\operatorname{rg} e^{tA} \subseteq \operatorname{dom}\left(A^\ell\right)$ for all $t > 0$ and $\ell \in \mathbb{N}_0$.*

Proof. Replace A with $A - \omega$, we may assume that $\omega_0(A) < 0$ and $\|\beta\mathcal{R}(\mathrm{i}\beta, A)\|$ is bounded for all $\beta \in \mathbb{R}$ of sufficiently large modulus. Hence, the integral representation formula for $t^{k-1}e^{tA}x$ in Proposition 12.1.2 holds for each $x \in X$, each $t > 0$ and each integer $k \geq 2$.

Now fix an integer $k \geq 2$ and observe that

$$A^{k-2}\mathcal{R}(\mathrm{i}\beta, A)^k = \left(-\mathrm{id}_X + \mathrm{i}\beta\mathcal{R}(\mathrm{i}\beta, A)\right)^{k-2} \mathcal{R}(\mathrm{i}\beta, A)^2,$$

and thus, $\|\beta^2\mathcal{R}(\mathrm{i}\beta, A)^k\|_{\operatorname{dom}(A^{k-2})\leftarrow X}$ is bounded for all $\beta \in \mathbb{R}$ of sufficiently large modulus. Hence, for each $x \in X$ and $t \geq 0$ the integral representation of $t^{k-1}e^{tA}x$ in Proposition 12.1.2 even converges as a Bochner integral in $\operatorname{dom}\left(A^{k-2}\right)$ and therefore, $e^{tA}x \in \operatorname{dom}\left(A^{k-2}\right)$ for all $t > 0$. Since $k \geq 2$ was arbitrary, the theorem is proved. □

The resolvent decay assumption from Theorem 12.1.4 is automatically satisfied if the semigroup generator is associated to a form on a Hilbert space (cf. Theorem 10.2.5).

Corollary 12.1.5. *Let the Hilbert spaces V, H and the sesquilinear form $\mathfrak{a}\colon V \times V \to \mathbb{C}$ satisfy the assumptions of Theorem 5.1.4. Then the semigroup generated by the operator $A\colon H \supseteq \operatorname{dom}(A) \to H$ associated to $\mathfrak{a}$ satisfies $\operatorname{rg} e^{tA} \subseteq \operatorname{dom}\left(A^\ell\right)$ for all $t > 0$ and $\ell \in \mathbb{N}_0$.*

Proof. By assumption, $\mathfrak{a}$ satisfies the ellipticity estimate

$$\operatorname{Re}\mathfrak{a}(v, v) + \mu\|v\|_H^2 \geq \delta\|v\|_V^2 \tag{12.1.1}$$

for some numbers $\mu \in \mathbb{R}$ and $\delta > 0$ and for all $v \in V$. After shifting the entire situation by a real number if necessary, we may assume that $\mu < 0$ and that $\omega_0(A) < 0$.

The idea is to first check that the ellipticity estimate (12.1.1) is, up to a change of δ, preserved if we rotate $\mathfrak{a}$ by a small angle. We use this to show that $\left\| \beta \mathcal{R}(\mathrm{i}\beta, A) \right\|$ is bounded on $\mathbb{R} \setminus \{0\}$. The claim then follows from Theorem 12.1.4.

The continuity of the form $\mathfrak{a}$ implies that $|\mathfrak{a}(v,v)| \le M \|v\|_V^2$ for a number $M \ge 0$ and all $v \in V$. Hence, if we choose $\theta > 0$ sufficiently close to 0, then

$$\operatorname{Re}\left(e^{-\mathrm{i}\theta}\mathfrak{a}(v,v)\right) + \mu \|v\|_H^2 \ge \delta \|v\|_V^2 - \left|\left(e^{-\mathrm{i}\theta} - 1\right)\right| |\mathfrak{a}(v,v)| \ge \frac{\delta}{2} \|v\|_V^2 \tag{12.1.2}$$

for all $v \in V$ and $\operatorname{Re}\left(\mathrm{i}e^{-\mathrm{i}\theta}\right) = \sin\theta > 0$; we will use the latter property at the end of the proof. The inequality (12.1.2) shows that the form $e^{-\mathrm{i}\theta}\mathfrak{a}$ also satisfies the assumptions of Theorem 5.1.4, with the same number $\mu < 0$ (and a different δ). One easily checks that the operator $e^{-\mathrm{i}\theta}A$ is associated to $e^{-\mathrm{i}\theta}\mathfrak{a}$, so Theorem 5.1.4(b) gives $\mathrm{s}(e^{-\mathrm{i}\theta}A) \le \mu < 0$ and

$$\left\| \mathcal{R}(\lambda, e^{-\mathrm{i}\theta}A) \right\| \le \frac{1}{\operatorname{Re}\lambda - \mu} \le \frac{1}{\operatorname{Re}\lambda}$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0 > \mu$. In particular for $\lambda := e^{-\mathrm{i}\theta}\mathrm{i}\beta$ with $\beta \in (0,\infty)$, this gives,

$$\left\| \mathcal{R}(\mathrm{i}\beta, A) \right\| = \left\| \mathcal{R}(e^{-\mathrm{i}\theta}\mathrm{i}\beta, e^{-\mathrm{i}\theta}A) \right\| \le \frac{1}{\operatorname{Re}\left(e^{-\mathrm{i}\theta}\mathrm{i}\beta\right)} = \frac{1}{\beta\sin\theta}.$$

Considering a negative θ sufficiently close to 0, yields a similar estimate for $\beta \in (-\infty, 0)$. ☐

Let us point out that the results in Theorem 12.1.4 and Corollary 12.1.5 are far from optimal. The decay assumption for the resolvent in the theorem implies – and is in fact to equivalent to – an even stronger regularity property of the semigroup, namely that it is **analytic**. Analyticity of a semigroup is a strong and useful property, and a thorough treatment could easily fill to chapters on its own. Since the concept is not strictly necessary for the remaining sections of the course, we refrain from studying analytic semigroups and instead refer the interested readers to the literature, e.g. to [EN00, Section II.4.a].

12.2 Eventual norm continuity

Definition 12.2.1 (Eventual norm continuity)**.** A C_0-semigroup $(e^{tA})_{t\ge0}$ on a Banach space X is called **eventually norm continuous** if there exists $t_0 \ge 0$ such that the mapping

$$[t_0, \infty) \to \mathcal{L}(X), \qquad t \mapsto e^{tA}$$

is continuous with respect to the operator norm.

The main reason why we are interested in eventually norm continuous semigroups is that their long-term behaviour is quite well understood. For instance, their spectral and growth bounds are always equal, even without any positivity assumption.

Theorem 12.2.2. *Let $(e^{tA})_{t\geq 0}$ be an eventually norm continuous C_0-semigroup on a complex Banach space X. Then the following properties hold:*

(a) $\mathrm{s}(A) = \omega_0(A)$.

(b) *If $\mathrm{s}(A) > -\infty$, then the supremum in the definition of the spectral bound is a maximum, i.e. there exists a $\mu \in \sigma(A)$ such that $\operatorname{Re}\mu = \mathrm{s}(A)$.*

Proof. We prove both claims together. Let $t_0 \geq 0$ such that $t \mapsto e^{tA}$ is operator norm continuous on $[t_0,\infty)$. There is no loss of generality in assuming $t_0 > 0$ and $\omega_0(A) = 0$ throughout the proof. We then need to show that A has a spectral value on $\mathrm{i}\mathbb{R}$.

A similar argument as in the proof of Proposition 11.2.2 shows that $\omega_0(A) = 0$ implies $\mathrm{r}(e^{t_0 A}) = 1$ (see Exercise 12.1 for details). Hence choose an angle $\theta \in \mathbb{R}$ such that $e^{\mathrm{i}\theta} \in \sigma(e^{t_0 A})$. As $e^{\mathrm{i}\theta} \in \partial\sigma(e^{t_0 A})$, it follows that $e^{\mathrm{i}\theta}$ is an approximate eigenvalue of $e^{t_0 A}$ (Exercise 5.2), i.e. there exists a sequence (x_n) in X with $\|x_n\| = 1$ for each n and such that $(e^{\mathrm{i}\theta} - e^{t_0 A})x_n \to 0$ as $n \to \infty$. For each index n, we choose a functional $x_n' \in X'$ of norm $\|x_n'\| = 1$ such that $\langle x_n', x_n\rangle \to 1$ as $n \to \infty$.

Consider, for each n, the map $f_n \in \mathrm{C}([0,t_0];\mathbb{C})$ given by

$$f_n(s) = \left\langle x_n', e^{-\mathrm{i}\frac{\theta}{t_0}s} e^{(s+t_0)A} x_n \right\rangle$$

for all $s \in [0,t_0]$. The sequence (f_n) is bounded in $\mathrm{C}([0,t_0];\mathbb{C})$, and it is equicontinuous since the map $t \mapsto e^{tA}$ is continuous with respect to the operator norm on $[t_0,\infty)$, as shown in the first step of the proof. By the Arezlà–Ascoli theorem, we may replace the sequences (f_n), (x_n), and (x_n') with subsequences and thus achieve that f_n converges in $\|\cdot\|_\infty$ to a function $f \in \mathrm{C}([0,t_0];\mathbb{C})$. The approximate eigenvector property of (x_n) implies that $f_n(0) \to e^{\mathrm{i}\theta}$, so $f(0) = e^{\mathrm{i}\theta}$.

Therefore, f is non-zero and hence, so is at least one Fourier coefficient of f, i.e. there exists an integer $\ell \in \mathbb{Z}$ such that $\int_0^{t_0} e^{-\mathrm{i}\frac{2\pi\ell}{t_0}s} f(s)\,\mathrm{d}s \neq 0$. Thus, $\int_0^{t_0} e^{-\mathrm{i}\frac{2\pi\ell}{t_0}s} f_n(s)\,\mathrm{d}s \not\to 0$, as $n \to \infty$ which in turn implies that

$$\int_0^{t_0} e^{s(A-\mathrm{i}\beta)} e^{t_0 A} x_n \,\mathrm{d}s \overset{n\to\infty}{\not\to} 0, \qquad \text{where} \qquad \beta := \frac{\theta + 2\pi\ell}{t_0}. \tag{12.2.1}$$

We complete the proof by showing that $\mathrm{i}\beta \in \sigma(A)$. Since $s \mapsto e^{sA}e^{t_0 A}x$ is a mild solution to the abstract Cauchy problem $\dot{u}(s) = Au(s)$, $u(0) = e^{t_0 A}x$, one has

$$\begin{aligned}(\mathrm{i}\beta - A)\int_0^{t_0} e^{s(A-\mathrm{i}\beta)} e^{t_0 A} x_n &= e^{t_0 A}x_n - e^{t_0(A-\mathrm{i}\beta)}e^{t_0 A}x_n \\ &= e^{-\mathrm{i}\theta}e^{t_0 A}\big(e^{\mathrm{i}\theta}x_n - e^{t_0 A}x_n\big) \to 0 \text{ as } n \to \infty.\end{aligned}$$

If $\mathrm{i}\beta \notin \sigma(A)$, we could multiply with $\mathcal{R}(\mathrm{i}\beta, A)$ and would obtain a contradiction to (12.2.1). ☐

Let us now give two sufficient conditions for a C_0-semigroup to be eventually norm continuous. The first condition is the type of smoothing property that we studied in detail in the previous section.

Proposition 12.2.3. *Let* $(e^{tA})_{t\ge 0}$ *be a* C_0*-semigroup on a Banach space* X *that satisfies* $\operatorname{rg} e^{t_0 A} \subseteq \operatorname{dom}(A)$ *for some* $t_0 \ge 0$*. Then* $(e^{tA})_{t\ge 0}$ *is eventually norm continuous.*

Proof. Note that, by the closed graph theorem, $e^{t_0 A}$ is continuous from X to $\operatorname{dom}(A)$, when the latter space is endowed with a graph norm of A. Fix a $\tau > t_0$ and set $M := \sup_{t\in[0,\tau]} \|e^t A\|$. For all $t_1, t_2 \in [t_0, \tau]$ with $t_1 \le t_2$ and all $x \in X$, Proposition 12.1.1 and the fundamental theorem of calculus imply

$$\begin{aligned} \|(e^{t_2 A} - e^{t_1 A})x\|_X &= \left\| \int_{t_1}^{t_2} A e^{sA} x \, \mathrm{d}s \right\|_X \\ &\le (t_2 - t_1) \underbrace{\|A\|_{X\leftarrow \operatorname{dom}(A)} \left\|e^{t_0 A}\right\|_{\operatorname{dom}(A)\leftarrow X} M}_{=:\widetilde{M}} \|x\| . \end{aligned}$$

This implies that $\left\|e^{t_2 A} - e^{t_1 A}\right\|_{X\leftarrow X} \le \widetilde{M} \, |t_2 - t_1|$. for all $t_1, t_2 \in [t_0, \tau]$. ☐

To give a first example of how the results from this and the previous section can be applied, we revisit an old friend, the Dirichlet Laplace operator. This is mainly a toy example in the present context, to demonstrate the previous results in a now-familiar situation. The following properties of Δ_{Dir} can – even under weaker assumptions – also be derived by other methods, for instance the spectral theorem for self-adjoint operators.

Example 12.2.4 (The Dirichlet Laplacian)**.** Let $\emptyset \ne \Omega \subseteq \mathbb{R}^n$ be open, bounded, and connected. Assume for the sake of simplicity that Ω has C^∞ boundary.[1]

Consider the Dirichlet Laplace operator $\Delta_{\mathrm{Dir}} \colon L^2(\Omega) \supseteq \operatorname{dom}(\Delta_{\mathrm{Dir}}) \to L^2(\Omega)$. One has $e^{t\Delta_{\mathrm{Dir}}} \to 0$ with respect to the operator norm as $t \to \infty$.

Proof. As shown in Example 7.3.8(b) one has $\mathrm{s}(\Delta_{\mathrm{Dir}}) < 0$. Moreover, as Δ_{Dir} is associated to a sesquilinear form $\mathfrak{a}$ that satisfies the assumptions of Theorem 5.1.4, we can apply Corollary 12.1.5 to conclude that $\operatorname{rg} e^{t\Delta_{\mathrm{Dir}}} \subseteq \operatorname{dom}(\Delta_{\mathrm{Dir}})$ for all $t > 0$. In particular, the semigroup is eventually norm continuous[2] according to Proposition 12.2.3 and hence, Theorem 12.2.2(a) shows that $\mathrm{s}(\Delta_{\mathrm{Dir}}) = \omega_0(\Delta_{\mathrm{Dir}})$. ☐

The second sufficient condition that we give for eventual norm continuity is that one of the semigroup operators be compact. We need the following elementary result from functional analysis.

Lemma 12.2.5. *Let* X *be a Banach space and let* $S, T \in \mathcal{L}(X)$*. Let* (S_j) *be a bounded net in* $\mathcal{L}(X)$ *that converges strongly to* S*, i.e.* $S_j x \to S$ *for all* $x \in X$*. If* T *is compact, then* $S_j T \to ST$ *in operator norm.*

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[1] In fact, we only need the smoothness assumption on Ω to apply the result in Example 7.3.8(b) that shows $\mathrm{s}(\Delta_{\mathrm{Dir}}) < 0$. As pointed out on several earlier occasions, the property $\mathrm{s}(\Delta_{\mathrm{Dir}}) < 0$ is also a consequence of Poincaré's inequality and is in fact true without any regularity assumptions on the boundary of Ω. Moreover, it does not require Ω to be bounded, either – it suffices if Ω is contained in a strip of finite width.

[2] In fact, a glance at the proof of Proposition 12.2.3 shows that the semigroup is even norm continuous on $(0,\infty)$ since the time $t_0 > 0$ in the Proposition can be chosen arbitrarily small.

Proof. Since the set $TB_{\leq 1}(0)$ is relatively compact, it is totally bounded, i.e. for every $\varepsilon > 0$ it can be covered by finitely many balls of radius ε. By using this property, the proof is now straightforward. ☐

Proposition 12.2.6. *Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a Banach space X such that $e^{t_0 A} \in \mathcal{L}(X)$ is compact for some $t_0 \geq 0$. Then $(e^{tA})_{t\geq 0}$ is eventually norm continuous.*

Proof. Let $t \in [t_0, \infty)$ and let (t_k) be a sequence in $[t_0, \infty)$ that converges to t. By the strong continuity of the semigroup, the sequence $(e^{(t_n - t_0)A})$ converges strongly to $e^{(t-t_0)A}$, and by the uniform boundedness principle the sequence is bounded. Since $e^{t_0 A}$ is compact, Lemma 12.2.5 implies that

$$e^{t_k A} = e^{(t_k - t_0)A} e^{t_0 A} \to e^{(t-t_0)A} e^{t_0 A} = e^{tA}$$

with respect to the operator norm. ☐

12.3 Spectral decomposition and convergence to equilibrium

In Chapter 11, we saw some sufficient conditions for a C_0-semigroup to converge to 0 in the operator norm. In the present section, we are interested in the conditions which ensure that the semigroup operators converge to a non-zero operator in the operator norm. To this end, we use that the spectral decomposition from Proposition 6.2.2 is compatible with C_0-semigroups in the following sense.

Proposition 12.3.1 (Spectral decomposition for semigroups)**.** *Let $(e^{tA})_{t\geq 0}$ be an eventually norm-continuous C_0-semigroup on a complex Banach space X. Assume that $\sigma_0 \subseteq \sigma(A)$ is compact, $\sigma(A) \setminus \sigma_0$ is closed, and let $P \in \mathcal{L}(X)$ be the spectral projection of A associated to σ_0 (Definition 6.2.3).*

(a) *For each $t > 0$, e^{tA} commutes with P and thus leaves* $\ker P$ *and* $\operatorname{rg} P$ *invariant.*

(b) *The restricted operator families $\left(e^{tA}|_{\operatorname{rg} P}\right)_{t\geq 0}$ and $\left(e^{tA}|_{\ker P}\right)_{t\geq 0}$ are C_0-semigroups on* $\operatorname{rg} P$ *and* $\ker P$*, respectively, and their generators are $A|_{\operatorname{rg} P}$ and $A|_{\ker P}$.*[3]

Proof. (a) One can easily derive from Proposition 6.2.2(a) that P commutes with $\mathcal{R}(\cdot, A)$ for every $\lambda \in \rho(A)$ and hence, P also commutes with the approximating operators A_n from formula (10.2.1) for all n. As e^{tA_n} is given as a series over powers of A_n for each $t \geq 0$, P also commutes with e^{tA_n} for all indices n and all $t \geq 0$. Since $e^{tA_n} \to e^{tA}$ strongly as $n \to \infty$, we finally conclude that P commutes with e^{tA} for each $t \geq 0$.

(b) Both families are obviously C_0-semigroups and the definition of semigroup generators (Definition 10.2.1(b)) readily gives the claimed equalities for the generators. ☐

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[3] We recall from Proposition 6.2.2 that $A|_{\operatorname{rg} P} \in \mathcal{L}(\operatorname{rg} P)$ and that the operator $A|_{\ker P}$ on $\ker P$ has the domain $\operatorname{dom}(A) \cap \ker P$.

Theorem 12.3.2 (Convergence to equilibrium)**.** *Let $(e^{tA})_{t\ge 0}$ be an eventually norm continuous C_0-semigroup on a complex Banach space X. Assume that $\mathrm{s}(A) = 0$ and that $\sigma(A)\cap \mathrm{i}\mathbb{R}$ consists only of poles of the resolvent. The following are equivalent:*

(i) *e^{tA} converges with respect to the operator norm to an operator $\mathcal{L}(X)$ as $t \to \infty$.*

(ii) *e^{tA} converges strongly to an operator in $\mathcal{L}(X)$ as $t \to \infty$.*

(iii) *$\sigma(A)\cap \mathrm{i}\mathbb{R} = \{0\}$ and 0 is a first order pole of $\mathcal{R}(\cdot, A)$.*

If those equivalent assertions are satisfied, then the operator $\lim_{t\to\infty} e^{tA}$ is the spectral projection of A associated to the spectral value 0.

Proof. Note that $\sigma(A)\cap \mathrm{i}\mathbb{R} \neq \emptyset$ by Theorem 12.2.2(b).

"(i) $\Rightarrow$ (ii)": This implication is obvious.

"(ii) $\Rightarrow$ (iii)": Assume that (iii) fails. Then precisely one of the following two cases occurs:

$\sigma(A)\cap \mathrm{i}\mathbb{R}$ contains a point $\mathrm{i}\beta \neq 0$: In this case, $\mathrm{i}\beta$ is a pole of $\mathcal{R}(\cdot, A)$ by assumption and hence an eigenvalue of A (Theorem 6.2.6(a)). Let $x \in \operatorname{dom}(A)$ be an eigenvector of A for the eigenvalue $\mathrm{i}\beta$. Then one readily checks that the mapping $u\colon [0,\infty) \to X$, $u(t) = e^{\mathrm{i}\beta t}x$ is a classical solution to the Cauchy problem $\dot u(t) = Au(t)$, $u(0) = x$, so $e^{tA}x = u(t) = e^{\mathrm{i}\beta t}x$ for all $t \ge 0$. Since $\beta \neq 0$, it follows that $e^{tA}x$ does not converge as $t \to \infty$.

$\sigma(A)\cap \mathrm{i}\mathbb{R} = \{0\}$, but the pole order of 0 is $p \ge 2$: Then one gets from Theorem 6.2.6(a) and (b) that 0 is an eigenvalue of A that is not semisimple. Hence, there exists a vector $x \in \ker A^2 \setminus \ker A$. It is now easy to check that the mapping $u\colon [0,\infty) \to X$, $u(t) = tAx + x$ is a classical solution to the Cauchy problem $\dot u(t) = Au(t)$, $u(0) = x$. Hence, $e^{tA}x = tAx + x$ for all $t \ge 0$. Since $Ax \neq 0$, this solution does not converge in X as $t \to \infty$.

"(iii) $\Rightarrow$ (i)": Let P denotes the spectral projection of A associated to the point 0. Since 0 is a first order pole of $\mathcal{R}(\cdot, A)$, Theorem 6.2.6(b) and (c) shows that $\operatorname{rg} P = \ker A$, so $A|_{\operatorname{rg} P}$ is the zero operator on $\operatorname{rg} P$. Hence, the semigroup acts as the identity on $\operatorname{rg} P$.

On the other hand, Proposition 6.2.2(b) shows that $\sigma(A|_{\ker P}) = \sigma(A) \setminus \{0\}$ and hence, every spectral value of $A|_{\ker P}$ has strictly negative real part. Obviously, the restriction of the semigroup to $\ker P$ is also eventually norm continuous, so it follows from Theorem 12.2.2(b) that $\mathrm{s}(A|_{\ker P}) < 0$ and from part (a) of the same theorem that spectral and growth bounds of $A|_{\ker P}$ coincide. Since this operator generates the restriction of the semigroup to $\ker P$, this restriction converges to 0 in operator norm as $t \to \infty$.

Additional property at the end of the theorem: From the proof of the implication (iii) $\Rightarrow$ (i), it follows that $\lim_{t\to\infty} e^{tA}$ is indeed the spectral projection of A associated to 0. ☐

As a simple example for Theorem 12.3.2, we discuss an application to the Neumann Laplace operator in Exercise 12.2.

12.4 A convergence result for eventually positive semigroups

In this section we demonstrate that eventual positivity does, under sufficiently strong assumptions on the generator, imply convergence to equilibrium as $t \to \infty$. While this is an interesting insight in its own right, it will also turn out to be very useful for the proof of the characterisations in the next chapter.

Theorem 12.4.1. *Let E be a complex Banach lattice. Let $(e^{tA})_{t\ge 0}$ be a C_0-semigroup on E that is real* [4]*, individually eventually positive with respect to* 0*, and eventually norm continuous. Assume that A has compact resolvent and that $\operatorname{s}(A) > -\infty$, and let P denote the spectral projection of A associated to $\operatorname{s}(A)$. The following are equivalent:*

(i) *The pole $\operatorname{s}(A)$ of $\mathcal{R}(\cdot, A)$ has order one.*

(ii) *$e^{t(A-\operatorname{s}(A))} \to P$ with respect to the operator norm as $t \to \infty$.*

For the proof of the theorem we need the following nice lemma from topological dynamics on the complex unit circle $\mathbb{T}$.

Lemma 12.4.2. *Let $m \in \mathbb{N}$ and let $\mu_1, \dots, \mu_m \in \mathbb{T}$. There exists a sequence of integers $1 \le \ell_k \to \infty$ such that $\mu_j^{\ell_k} \to 1$ as $k \to \infty$ for all $j \in \{1, \dots, m\}$.*

The lemma is, in fact, a special case of the following more general result on compact groups. Just apply Proposition 12.4.3 to the group $\mathbb{T}^m$ (endowed with the pointwise multiplication) to obtain Lemma 12.4.2.

Proposition 12.4.3. *Let G be a group with neutral element 1 and, at the same time, a compact metric space. Assume that the group operation $G \times G \to G$, $(g, h) \to gh$ is continuous.*[5] *For each $g \in G$ there exists a sequence of integers $1 \le \ell_k \to \infty$ such that $g^{\ell_k} \to 1$ as $k \to \infty$.*

Proof. Fix $g \in G$. We say that a set $S \subseteq G$ is **left invariant under** g if $gS \subseteq S$. As G is compact, the intersection of any chain of non-empty closed subsets of G is non-empty. Hence, it follows from Zorn's lemma that, among all subsets of G that are non-empty, closed, and left invariant under g, there exists at least one – let us call it M – that is minimal under set inclusion.

Choose an arbitrary element $h \in M$. For every $\ell_0 \in \mathbb{N}$ the closure of $\{g^\ell h : \ell \ge \ell_0\}$ is a non-empty, closed subset of M that is left invariant under g. So by the minimality of M, the closure of $\{g^\ell h : \ell \ge \ell_0\}$ equals M; in particular, it contains h. Thus, one can construct a strictly increasing sequence (ℓ_k) in $\mathbb{N}$ such that $g^{\ell_k} h \to h$ as $k \to \infty$. Finally, we multiply with h^{-1} from the right to obtain the conclusion of the proposition. □

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[4] This means that e^{tA} is real for each $t \ge 0$.

[5] By a theorem of Ellis (see for instance [EFHN15, Theorem G.12]) this is equivalent to the formally weaker property that the multiplication on G is separately continuous in each variable. Moreover, it implies that the map $G \to G$, $g \mapsto g^{-1}$ is continuous (in other words, G is a compact topological group).

We note in passing that the proposition remains true for compact topological groups that are not metrisable (but whose topology is Hausdorff) if one allows (ℓ_k) to be a net rather than a sequence.

Proof of Theorem 12.4.1. We may assume throughout the proof that $\mathrm{s}(A) = 0$. Observe that the individual eventual positivity then implies $0 \in \sigma(A)$ according to Theorem 11.4.2. Choose $t_0 \geq 0$ such that $t \mapsto e^{tA}$ is operator norm continuous on $[t_0, \infty)$.

"(ii) ⇒ (i)": This implication follows from Theorem 12.3.2.

"(i) ⇒ (ii)": We proceed in several steps.

We show that $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is finite: Since A has compact resolvent, all spectral values of A are isolated (Theorem 6.2.9(b)) so it suffices to prove that $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is bounded. Assume to the contrary that there exists $(\mathrm{i}\beta_n) \subseteq \sigma(A) \cap \mathrm{i}\mathbb{R}$ such that $|\beta_n| \to \infty$.

Since A has compact resolvent, each $\mathrm{i}\beta_n$ is an eigenvalue of A (Theorem 6.2.9(a)), say with normalised eigenvector $x_n \in E$. Since the orbits of the semigroups are solutions of the corresponding Cauchy problem, $e^{tA}x_n = e^{\mathrm{i}\beta_n t}x_n$ for all $t \geq 0$ and all $n \in \mathbb{N}$. Hence, the set $\{t \mapsto e^{tA}x_n : n \in \mathbb{N}\} \subseteq \mathrm{C}([t_0, t_0 + 1], X)$ is not equicontinuous as $|\beta_n| \to \infty$. This contradicts the norm continuity of $t \mapsto e^{tA}$ on $[t_0, t_0 + 1]$.

We show each $\lambda \in \sigma(A) \cap \mathrm{i}\mathbb{R}$ is a first order pole of $\mathcal{R}(\cdot, A)$: As A has compact resolvent, each of its spectral values is a pole of the resolvent (Theorem 6.2.9(b)). It follows from Exercise 11.3(c) that the pole order of all spectral values on the imaginary axis is dominated by the pole order of the spectral value 0. Since the latter is equals 1 by assumption, the claim of this step is proved.

We show that the spectral projection $Q \in \mathcal{L}(X)$ of $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is positive: We denote the spectral values of A in $\mathrm{i}\mathbb{R}$ by $\mathrm{i}\beta_1, \dots, \mathrm{i}\beta_m$, and let $Q_1, \dots, Q_m$ denote the associated spectral projections. Each $\mathrm{i}\beta_j$ is a first order pole of $\mathcal{R}(\cdot, A)$ according to the previous step. Thus, Theorem 6.2.6(b) and (c) show that $\operatorname{rg} Q_j = \ker(\mathrm{i}\beta_j - A)$ for each $j \in \{1, \dots, m\}$. Hence, for each such j and each $t \geq 0$, the semigroup operator e^{tA} acts on $\operatorname{rg} Q_j$ as the multiplication with the scalar $e^{\mathrm{i}\beta_j t}$. The contour integral formula for spectral projections in Proposition 6.2.2 implies that $Q = Q_1 + \dots + Q_m$ and $Q_i Q_j = 0$ whenever $i \neq j$. Thus $\operatorname{rg} Q = \bigoplus_{j=1}^m \ker(\mathrm{i}\beta_j - A)$. According to Lemma 12.4.1, there exists a sequence $(\ell_k) \to \infty$ in $\mathbb{N}$ such that $e^{\mathrm{i}\beta_j \ell_k} \to 1$ for each $j \in \{1, \dots, m\}$ as $k \to \infty$. This implies that $e^{\ell_k A}x \to x$ as $k \to \infty$ for each $x \in \operatorname{rg} Q$.

On the other hand, the generator $A|_{\ker Q}$ of the restricted semigroup on $\ker Q$ (Proposition 12.3.1(b)), has no spectral value on $\mathrm{i}\mathbb{R}$ according to the spectral splitting property of the spectral projection Q (Proposition 6.2.2(b)). It follows from Theorem 12.2.2 that $\omega_0(A|_{\ker Q}) = \mathrm{s}(A|_{\ker Q}) < 0$ and hence, the semigroup on $\ker Q$ converges to 0 in operator norm as $t \to \infty$.

In summary, one has $e^{\ell_k A}x \to Qx$ as $k \to \infty$ for each $x \in E$. Since the semigroup is individually eventually positive, it follows that $Q \geq 0$, as claimed.

We show that $\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}$ and derive (ii): Since Q is a positive projection on a complex Banach lattice, it follows that its range $\operatorname{rg} Q$ is a complex Banach lattice in its own right[6] (Exercise 4.4(c)). Observe that the eigenvalues $\mathrm{i}\beta_1, \dots, \mathrm{i}\beta_m$ of A have

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[6] More precisely, the order on the real part of $\operatorname{rg} Q$ is inherited from the order on the real part of E, but the lattice operators on (the real part of) $\operatorname{rg} Q$ might be different from those on (the real part of) E.

finite algebraic multiplicity, since A has compact resolvent (Theorem 6.2.9(a)). Therefore, all the operators $Q_1, \dots, Q_m$ have finite-dimensional range and hence, so does Q. Hence $\operatorname{rg} Q$ is a finite-dimensional complex Banach lattice. One can show that every finite-dimensional real Banach lattice is isomorphic to $\mathbb{R}^n$ for some n (see e.g. [Sch74, Corollary 1 to Theorem II.3.9]) and from this it follows that every finite-dimensional complex Banach lattice is isomorphic to $\mathbb{C}^n$. As we assumed the semigroup $(e^{tA})_{t\geq 0}$ to be real, so is its generator A. Hence, we can identify its restriction $A|_{\operatorname{rg} Q}$ with a real matrix in $\mathbb{R}^{n\times n}$ that generates an eventually positive semigroup.

The finite-dimensional result Theorem 2.2.3(b) now implies that $\sigma(A|_{\operatorname{rg} Q}) = \{0\}$. Since $\sigma(A|_{\operatorname{rg} Q}) = \sigma(A) \cap i\mathbb{R}$ by the choice of Q and by Proposition 6.2.2(b), it follows that $\sigma(A) \cap i\mathbb{R} = \{0\}$. Observe that this also implies $Q = P$. To conclude, the operator norm convergence $e^{tA} \to P$ as $t \to \infty$ follows from Theorem 12.3.2(iii).

$\square$

Exercises for Chapter 12

Exercise 12.1. Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a complex Banach space X and let $t_0 > 0$. Set $e^{-\infty} := 0$. Show that

$$e^{t_0\omega_0(A)} = \mathrm{r}(e^{t_0 A}).$$

Exercise 12.2. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open, bounded, and connected, and assume that Ω has C^∞ boundary. Let $\Delta_{\mathrm{Neu}} : L^2(\Omega) \supseteq \operatorname{dom}(\Delta_{\mathrm{Neu}}) \to L^2(\Omega)$ be the Neumann Laplace operator introduced in Example 9.3.3. Show that

$$e^{t\Delta_{\mathrm{Neu}}} \to \frac{1}{|\Omega|} \mathbb{1} \otimes \mathbb{1},$$

with respect to the operator norm as $t \to \infty$, where $|\Omega|$ is the Lebesgue measure of Ω.

Hint: Theorem 12.3.2.

Exercise 12.3 (A refinement of Example 12.2.4)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open, bounded, and connected, and assume that Ω has C^∞ boundary. Let $\Delta_{\mathrm{Dir}} : L^2(\Omega) \supseteq \operatorname{dom}(\Delta_{\mathrm{Dir}}) \to L^2(\Omega)$ be the Dirichlet Laplace operator, and set $\lambda_0 := \mathrm{s}(\Delta_{\mathrm{Dir}})$.

In Example 7.3.8, we proved that $\lambda_0 < 0$ and $\ker(\lambda_0 - \Delta_{\mathrm{Dir}})$ is spanned by a positive function v. Assume that v is normalised in $L^2(\Omega)$, i.e. $\|v\|_{L^2(\Omega)} = 1$. Prove that

$$e^{-\lambda_0 t} e^{t\Delta_{\mathrm{Dir}}} \to v \otimes v$$

in operator norm as $t \to \infty$.

Exercise 12.4 (Modes of convergence)**.** Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a Banach space X and assume that e^{tA} converges strongly to an operator $P \in \mathcal{L}(X)$ as $t \to \infty$.

(a) Prove that $\operatorname{rg} P$ is the space of all fixed vectors of the semigroup (i.e. the space of all those $x \in X$ that satisfy $e^{tA}x = x$ for all $t \geq 0$.)

(b) Show that P is a projection that commutes with e^{tA} for every $t \geq 0$.

(c) Assume now that the convergence $e^{tA} \to P$ as $t \to \infty$ even takes place with respect to the operator norm. Prove that there exist numbers $M \geq 0$ and $\omega < 0$ such that

$$\left\| e^{tA} - P \right\| \leq M e^{\omega t} \qquad \text{for all } t \geq 0.$$

(d) Under the assumption of (c), show that 0 is isolated spectral value of A and a first order pole of $\mathcal{R}(\cdot, A)$.

(e) Find an example where the convergence of e^{tA} to P as $t \to \infty$ is only strong but not in operator norm.

Exercise 12.5. Find an example of a Banach space X and a C_0-semigroup $(e^{tA})_{t\geq 0}$ on X such that $t \mapsto e^{tA}$ is operator norm continuous on $[1,\infty)$ but not on $(0,\infty)$.

Notes for Chapter 12

The results of this chapter are standard material in C_0-semigroup theory as it can, for instance, be found in [EN00] and [ABHN11]. We adjusted the presentation to pave an efficient way to the results that we will need for the next chapter.

The inverse Laplace transform

Proposition 12.1.2 – which can be interpreted as an inversion theorem for the Laplace transform, in the sense that it recovers the semigroup from the resolvent – critically assumes $k \geq 2$, since for $k = 1$ the resolvent decay is insufficient to ensure that integral exists in the Bochner sense (consider the space $X = \mathbb{C}$ to see that one cannot expect $\mathcal{R}(\mathrm{i}\beta, A)x$ to decay faster than $\frac{1}{|\beta|}$ as $\left|\beta\right| \to \infty$). However, under appropriate assumptions the integral still converges as an improper Bochner integral, i.e. one has

$$e^{tA}x = \frac{1}{2\pi} \lim_{n\to\infty} \int_{-n}^{n} e^{\mathrm{i}t\omega} \mathcal{R}(i\omega, A)x \, \mathrm{d}\omega$$

if $x \in \operatorname{dom}(A)$. This follows from integration by parts, see [ABHN11, Proposition 3.12.1] for details. If the underlying Banach space X is “sufficiently good”, the same formula even holds for all $x \in X$, see [ABHN11, Theorem 3.2.12].

Deriving Theorem 12.1.4 from the inverse Laplace (or Fourier) formula in Proposition 12.1.2 seems uncommon, though it is certainly unsurprising those working at the intersection of harmonic analysis and semigroup theory. A drawback of this approach is that it does not – at least not without additional effort – give the optimal result, namely analyticity of the generated semigroup (see the discussion at the end of Section 12.1).

Semigroups that map into $\operatorname{dom}(A)$

It is worthwhile to point out that there are C_0-semigroups $(e^{tA})_{t\geq 0}$ such that $\operatorname{rg} e^{tA} \subseteq \operatorname{dom}(A)$ for all $t > 0$, but not $\operatorname{rg} e^{tA} \subseteq \operatorname{dom}\left(A^2\right)$ – although we have not presented any methods in this chapter to identify such semigroups, since Theorems 12.1.4 and Corollary 12.1.5 yield the much stronger conclusions $e^{tA}X \subseteq \operatorname{dom}\left(A^{\ell}\right)$ for all $\ell \in \mathbb{N}_0$.

Further more one can find examples where indeed $\operatorname{rg} e^{t_0 A} \subseteq \operatorname{dom}(A)$ for some $t_0 > 0$ (and hence also for all $t \geq t_0$) but not necessarily for $t < t_0$. Due to the differentiability property in Proposition 12.1.1 such semigroups are called **eventually differentiable**.

Quite a lot can be said about eventual differentiability, and other eventual regularity properties of operators families that are not semigroups; see [Per22].

Eventual norm continuity

The proof of Theorem 12.2.2 is essentially taken from [EN00, Lemma IV.3.9]. As explained in this reference, the proof actually shows more, namely a characterisation of when the spectral mapping theorem holds for the approximate point spectrum of a semigroup generator. For more spectral information on eventually norm continuous semigroups we refer to [EN00, Section II.4.c].

Spectral decomposition and convergence to equilibrium

Spectral decomposition is a classical technique to study the long-term behaviour of C_0-semigroups, in particular in the operator norm. It is particularly well-suited to situations where the semigroup generator A has compact resolvent and can then often replace arguments based on the spectral theorem if the underlying space X is not a Hilbert space or A is not self-adjoint.

Other well-known technqiues include the so-called **Jacobs–de Leeuw–Glicksberg decomposition** that is based on an ingenious application of the theory of (semi)topological semigroups to functional analysis (see e.g. [EN00, Section V.2]), the theory of vector-valued Laplace transforms that is detailed in [ABHN11], and the concept of **constrictors** of semigroups that is described in [Eme07, Section 1.3].

Convergence of eventually positive semigroups

Theorem 12.4.1 is just one example of a variety of results about the long-term behaviour of eventually positive semigroups that can, in particular, be found in [AG21, Aro25].

Versions of Lemma 12.4.2 are quite prevalent in the literature that relate positivity properties to convergence to equilibrium. The ad hoc proof of Proposition 12.4.3, which can be formulated without introducing any further concepts from topological dynamics, is taken from [GG25, Lemma 4.3].

Bibliography

[AB06] Charalambos D. Aliprantis and Owen Burkinshaw. *Positive operators*. Berlin: Springer, reprint of the 1985 original edition, 2006.

[ABHN11] Wolfgang Arendt, Charles J. K. Batty, Matthias Hieber, and Frank Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2011.

[AE01] Herbert Amann and Joachim Escher. *Analysis III*. Grundstud. Math. Basel: Birkhäuser, 2001.

[AF03] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[AG21] Sahiba Arora and Jochen Glück. Spectrum and convergence of eventually positive operator semigroups. *Semigroup Forum*, 103(3):791–811, 2021.

[AG22a] Sahiba Arora and Jochen Glück. An operator theoretic approach to uniform (anti-)maximum principles. *J. Differential Equations*, 310:164–197, 2022.

[AG22b] Sahiba Arora and Jochen Glück. Stability of (eventually) positive semigroups on spaces of continuous functions. *C. R., Math., Acad. Sci. Paris*, 360:771–775, 2022.

[AG23] Sahiba Arora and Jochen Glück. A characterization of the individual maximum and anti-maximum principle. *Math. Z.*, 305(2):Paper No. 24, 17, 2023.

[AG24] Sahiba Arora and Jochen Glück. Irreducibility of eventually positive semigroups. *Stud. Math.*, 276(2):99–129, 2024.

[AGG$^+$86] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck. *One-parameter semigroups of positive operators*, volume 1184 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[Akh18] Khalid Akhlil. Locality and domination of semigroups. *Result. Math.*, 73(2):11, 2018. Id/No 59.

[AN09] Wolfgang Arendt and Robin Nittka. Equivalent complete norms and positivity. *Arch. Math.*, 92(5):414–427, 2009.

[Are06] Wolfgang Arendt. Heat kernels: ISEM 2005/6, 2006. Available at `https://www.uni-ulm.de/fileadmin/website_uni_ulm/mawi.inst.020/arendt/downloads/internetseminar.pdf`.

[Aro25] Sahiba Arora. Eventually positive semigroups: spectral and asymptotic analysis. *Semigroup Forum*, 110(2):263–295, 2025.

[AT07] Charalambos D. Aliprantis and Rabee Tourky. *Cones and duality*, volume 84 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2007.

[AU23] Wolfgang Arendt and Karsten Urban. *Partial differential equations. An introduction to analytical and numerical methods. Translated from the German by James B. Kennedy*, volume 294 of *Grad. Texts Math.* Cham: Springer, 2023.

[BKFR17] András Bátkai, Marjeta Kramar Fijavž, and Abdelaziz Rhandi. *Positive operator semigroups*, volume 257 of *Operator Theory: Advances and Applications.* Birkhäuser/Springer, Cham, 2017. From finite to infinite dimensions. With a foreword by Rainer Nagel and Ulf Schlotterbeck.

[Bou03] Nicolas Bourbaki. *Elements of mathematics. Algebra II. Chapters 4–7. Transl. from the French by P. M. Cohn and J. Howie.* Berlin: Springer, reprint of the 1990 English translation edition, 2003.

[Bou07] Nicolas Bourbaki. *Éléments de mathématique. Algèbre. Chapitres 4 à 7.* Berlin: Springer, reprint of the 1981 original edition, 2007.

[BP94] Abraham Berman and Robert J. Plemmons. *Nonnegative matrices in the mathematical sciences*, volume 9 of *Class. Appl. Math.* Philadelphia, PA: SIAM, 1994.

[BR84] Charles J. K. Batty and Derek W. Robinson. Positive one-parameter semigroups on ordered Banach spaces. *Acta Appl. Math.*, 2:221–296, 1984.

[Bra61] Alfred Brauer. On the characteristic roots of power-positive matrices. *Duke Math. J.*, 28:439–445, 1961.

[Bre11] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[BY84] Jonathan M. Borwein and David T. Yost. Absolute norms on vector lattices. *Proc. Edinb. Math. Soc., II. Ser.*, 27:215–222, 1984.

[CD13] Alexander P. Campbell and Daniel Daners. Linear algebra via complex analysis. *Amer. Math. Monthly*, 120(10):877–892, 2013.

[COS95] R. W. Cross, M. I. Ostrovskij, and V. V. Shevchik. Operator ranges in Banach spaces. I. *Math. Nachr.*, 173:91–114, 1995.

[Cro80] R. W. Cross. On the continuous linear image of a Banach space. *J. Aust. Math. Soc., Ser. A*, 29:219–234, 1980.

[CS00] Philippe Clément and Guido Sweers. Uniform anti-maximum principles. *J. Differ. Equations*, 164(1):118–154, 2000.

[CS01] Philippe Clément and Guido Sweers. Uniform anti-maximum principle for polyharmonic boundary value problems. *Proc. Am. Math. Soc.*, 129(2):467–474, 2001.

[Dal05] Anna Dall'Acqua. *Higher Order Elliptic Problems and Positivity*. PhD thesis, Delft University of Technology, 2005.

[Dav80] E. B. Davies. *One-parameter semigroups*, volume 15 of *Lond. Math. Soc. Monogr.* Academic Press, London, 1980.

[DG17] Daniel Daners and Jochen Glück. The role of domination and smoothing conditions in the theory of eventually positive semigroups. *Bull. Aust. Math. Soc.*, 96(2):286–298, 2017.

[DG18] Daniel Daners and Jochen Glück. Towards a perturbation theory for eventually positive semigroups. *J. Operator Theory*, 79(2):345–372, 2018.

[DGK16a] Daniel Daners, Jochen Glück, and James B. Kennedy. Eventually and asymptotically positive semigroups on Banach lattices. *J. Differential Equations*, 261(5):2607–2649, 2016.

[DGK16b] Daniel Daners, Jochen Glück, and James B. Kennedy. Eventually positive semigroups of linear operators. *J. Math. Anal. Appl.*, 433(2):1561–1593, 2016.

[DL00] Robert Dautray and Jacques-Louis Lions. *Mathematical analysis and numerical methods for science and technology. Volume 2: Functional and variational methods. With the collaboration of Michel Artola, Marc Authier, Philippe Bénilan, Michel Cessenat, Jean-Michel Combes, Hélène Lanchon, Bertrand Mercier, Claue Wild, Claude Zuily. Transl. from the French by Ian N. Sneddon.* Berlin: Springer, 2nd printing edition, 2000.

[DMS05] Anna Dall'Acqua, Christian Meister, and Guido Sweers. Separating positivity and regularity for fourth order Dirichlet problems in 2d-domains. *Analysis (München)*, 25(3):205–261, 2005.

[DPZ14] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*, volume 152 of *Encycl. Math. Appl.* Cambridge: Cambridge University Press, 2nd ed. edition, 2014.

[Dyn56] E. B. Dynkin. Markov processes and semi-groups of operators. *Teor. Veroyatn. Primen.*, 1:25–37, 1956.

[Dyn65] Evgeniĭ B. Dynkin. *Markov processes. Vols. I, II. Translated with the authorization and assistance of the author by J. Fabius, V. Greenberg, A. Maitra and G. Majone.*, volume 121/122 of *Grundlehren Math. Wiss.* Springer, Cham, 1965.

[EFHN15] Tanja Eisner, Bálint Farkas, Markus Haase, and Rainer Nagel. *Operator theoretic aspects of ergodic theory*, volume 272 of *Grad. Texts Math.* Cham: Springer, 2015.

[Eme07] Eduard Yu. Emel'yanov. *Non-spectral asymptotic analysis of one-parameter operator semigroups*, volume 173 of *Oper. Theory: Adv. Appl.* Basel: Birkhäuser, 2007.

[EN00] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[EN06] Klaus-Jochen Engel and Rainer Nagel. *A short course on operator semigroups.* Universitext. New York, NY: Springer, 2006.

[ES08] Abed Elhashash and Daniel B. Szyld. On general matrices having the Perron-Frobenius property. *Electron. J. Linear Algebra*, 17:389–413, 2008.

[ES09] Abed Elhashash and Daniel B. Szyld. Two characterizations of matrices with the Perron-Frobenius property. *Numer. Linear Algebra Appl.*, 16(11-12):863–869, 2009.

[Eva10] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2nd ed. edition, 2010.

[Fel52] Wiliam Feller. The parabolic differential equations and the associated semigroups of transformation. *Ann. Math. (2)*, 55:468–519, 1952.

[Fen98] Gero Fendler. On dilations and transference for continuous one-parameter semigroups of positive contractions on $\mathcal{L}^p$-spaces. *Ann. Univ. Sarav., Ser. Math.*, 9(1):1–97, 1998.

[Fra11] L. Edward Fraenkel. *An introduction to maximum principles and symmetry in elliptic problems*, volume 128 of *Camb. Tracts Math.* Cambridge: Cambridge University Press, reprint of the 2000 hardback edition edition, 2011.

[Fri78] Shmuel Friedland. On an inverse problem for nonnegative and eventually nonnegative matrices. *Israel J. Math.*, 29(1):43–60, 1978.

[Fro08] Georg Frobenius. Über Matrizen aus positiven Elementen. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, pages 471–476, 1908.

[Fro09] Georg Frobenius. Über Matrizen aus positiven Elementen. II. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, pages 514–518, 1909.

[Fro12] Georg Frobenius. Über Matrizen aus nicht negativen Elementen. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, pages 456–477, 1912.

[GG25] Jochen Glück and Ulrich Groh. A note on the positivity of inverse operators acting on $C^\star$-algebras. *Linear Algebra Appl.*, 708:337–354, 2025.

[GGK90] Israel Gohberg, Seymour Goldberg, and Marinus A. Kaashoek. *Classes of linear operators. Vol. I*, volume 49 of *Oper. Theory: Adv. Appl.* Basel etc.: Birkhäuser Verlag, 1990.

[GGS10] Filippo Gazzola, Hans-Christoph Grunau, and Guido Sweers. *Polyharmonic boundary value problems*, volume 1991 of *Lecture Notes in Mathematics*. Berlin: Springer, 2010.

[GH23] Jochen Glück and Julian Hölz. Eventual cone invariance revisited. *Linear Algebra Appl.*, 675:274–293, 2023.

[Glü16] Jochen Glück. *Invariant Sets and Long Time Behaviour of Operator Semigroups*. PhD thesis, Ulm University, 2016.

[Glü17] Jochen Glück. Towards a Perron-Frobenius theory for eventually positive operators. *J. Math. Anal. Appl.*, 453(1):317–337, 2017.

[Gol85] Jerome A. Goldstein. *Semigroups of linear operators and applications*. Oxford Math. Monogr. Oxford University Press, Oxford, 1985.

[GR10] Hans-Christoph Grunau and Frédéric Robert. Positivity and almost positivity of biharmonic Green's functions under Dirichlet boundary conditions. *Archive for Rational Mechanics and Analysis*, 195(3):865–898, Mar 2010.

[Gra14] Loukas Grafakos. *Modern Fourier analysis*, volume 250 of *Grad. Texts Math.* New York, NY: Springer, 3rd ed. edition, 2014.

[Gri11] Pierre Grisvard. *Elliptic problems in nonsmooth domains*, volume 69 of *Class. Appl. Math.* Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), reprint of the 1985 hardback ed. edition, 2011.

[Gru09] Gerd Grubb. *Distributions and operators*, volume 252 of *Grad. Texts Math.* New York, NY: Springer, 2009.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[GW20] Jochen Glück and Martin R. Weber. Almost interior points in ordered Banach spaces and the long-term behaviour of strongly positive operator semigroups. *Stud. Math.*, 254(3):237–263, 2020.

[Hal13] Brian C. Hall. *Quantum theory for mathematicians*, volume 267 of *Grad. Texts Math.* New York, NY: Springer, 2013.

[Haw08] Thomas Hawkins. Continued fractions and the origins of the Perron-Frobenius theorem. *Arch. Hist. Exact Sci.*, 62(6):655–717, 2008.

[Hen05] Dan Henry. *Perturbation of the boundary in boundary-value problems of partial differential equations. With editorial assistance from Jack Hale and Antônio Luiz Pereira*, volume 318 of *Lond. Math. Soc. Lect. Note Ser.* Cambridge: Cambridge University Press, 2005.

[Hil48] Einar Hille. *Functional analysis and semi-groups*, volume 31 of *Colloq. Publ., Am. Math. Soc.* American Mathematical Society (AMS), Providence, RI, 1948.

[HK23] Gerd Herzog and Peer C. Kunstmann. Eventually positive elements in ordered Banach algebras. *Commentat. Math. Univ. Carol.*, 64(3):321–330, 2023.

[HK24] Gerd Herzog and Peer Kunstmann. A Perron-Frobenius type result in Banach algebras via asymptotic closeness to a cone. *Positivity*, 28(3):10, 2024. Id/No 45.

[HvNVW16] Tuomas Hytönen, Jan van Neerven, Mark Veraar, and Lutz Weis. *Analysis in Banach spaces. Volume I. Martingales and Littlewood-Paley theory*, volume 63 of *Ergeb. Math. Grenzgeb., 3. Folge.* Cham: Springer, 2016.

[JT04] Charles R. Johnson and Pablo Tarazaga. On matrices with Perron-Frobenius properties and some negative entries. *Positivity*, 8(4):327–338, 2004.

[Kas17] Michael Kasigwa. *Eventual Cone Invariance.* ProQuest LLC, Ann Arbor, MI, 2017. Thesis (Ph.D.)–Washington State University.

[Ken94] Carlos E. Kenig. *Harmonic analysis techniques for second order elliptic boundary value problems: dedicated to the memory of Professor Antoni Zygmund*, volume 83 of *Reg. Conf. Ser. Math.* Providence, RI: American Mathematical Society, 1994.

[Kes17] Srinivasan Kesavan. A note on the grand theorems of functional analysis. *Math. Newsl., Ramanujan Math. Soc.*, 27(3):188–191, 2017.

[Kes21] Srinivasan Kesavan. The grand theorems of functional analysis revisited: a Baire-free approach. *Math. Newsl., Ramanujan Math. Soc.*, 31(3):89–93, 2021.

[KLS89] Mark A. Krasnosel'skii, Evgeniĭ A. Lifshits, and Mark V. Sobolev. *Positive linear systems. - The method of positive operators - Transl. from the Russian by Jürgen Appell*, volume 5 of *Sigma Ser. Appl. Math.* Berlin: Heldermann-Verlag, 1989.

[KT17] Michael Kasigwa and Michael J. Tsatsomeros. Eventual cone invariance. *Electron. J. Linear Algebra*, 32:204–216, 2017.

[KvG19] Anke Kalauch and Onno van Gaans. *Pre-Riesz spaces*, volume 66 of *De Gruyter Expo. Math.* Berlin: De Gruyter, 2019.

[Leo09] Giovanni Leoni. *A first course in Sobolev spaces*, volume 105 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2009.

[Lot68] Heinrich P. Lotz. Über das Spektrum positiver Operatoren. *Math. Z.*, 108:15–32, 1968.

[Lue82] Jesper Luetzen. *The prehistory of the theory of distributions*, volume 7 of *Stud. Hist. Math. Phys. Sci.* Springer-Verlag, New York, NY, 1982.

[Lun13] Alessandra Lunardi. *Analytic semigroups and optimal regularity in parabolic problems.* Mod. Birkhäuser Class. Basel: Birkhäuser, reprint of the 1995 hardback ed. edition, 2013.

[LZ71] Wilhelmus A. J. Luxemburg and Adriaan C. Zaanen. *Riesz spaces. Vol. I*, volume 1 of *North-Holland Math. Libr.* Elsevier (North-Holland), Amsterdam, 1971.

[Mac00] Charles R. MacCluer. The many proofs and applications of Perron's theorem. *SIAM Review*, 42(3):487–498, 2000.

[Maz11] Vladimir G. Maz'ya. *Sobolev spaces. With applications to elliptic partial differential equations. Transl. from the Russian by T. O. Shaposhnikova*, volume 342 of *Grundlehren Math. Wiss.* Berlin: Springer, 2nd revised and augmented ed. edition, 2011.

[MN91] Peter Meyer-Nieberg. *Banach lattices.* Universitext. Springer-Verlag, Berlin, 1991.

[MS64] Norman G. Meyers and James Serrin. $H = W$. *Proc. Natl. Acad. Sci. USA*, 51:1055–1056, 1964.

[MST99] Gustavo A. Muñoz, Yannis Sarantopoulos, and Andrew Tonge. Complexifications of real Banach spaces, polynomials and multilinear maps. *Stud. Math.*, 134(1):1–33, 1999.

[MW74] Günter Mittelmeyer and Manfred Wolff. Über den Absolutbetrag auf komplexen Vektorverbänden. *Math. Z.*, 137:87–92, 1974.

[Nou06] Dimitrios Noutsos. On Perron-Frobenius property of matrices having some negative entries. *Linear Algebra Appl.*, 412(2-3):132–153, 2006.

[NT08] Dimitrios Noutsos and Michael J. Tsatsomeros. Reachability and holdability of nonnegative states. *SIAM J. Matrix Anal. Appl.*, 30(2):700–712, 2008.

[Ouh05] El Maati Ouhabaz. *Analysis of heat equations on domains*, volume 31 of *London Mathematical Society Monographs Series.* Princeton University Press, Princeton, NJ, 2005.

[Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Appl. Math. Sci.* Springer, Cham, 1983.

[Per07a] Oskar Perron. Grundlagen für eine theorie des jacobischen kettenbruchalgorithmus. *Math. Ann.*, 64(1):1–76, 1907.

[Per07b] Oskar Perron. Zur Theorie der Matrices. *Math. Ann.*, 64(2):248–263, 1907.

[Per22] Marco Peruzzetto. On eventual regularity properties of operator-valued functions. *Stud. Math.*, 265(2):141–176, 2022.

[Pie07] Albrecht Pietsch. *History of Banach spaces and linear operators.* Birkhäuser Boston, Inc., Boston, MA, 2007.

[PS07] Patrizia Pucci and James Serrin. *The maximum principle*, volume 73 of *Prog. Nonlinear Differ. Equ. Appl.* Basel: Birkhäuser, 2007.

[Pul15] Ludwig Pulst. *Dominance of positivity of the Green's function associated to a perturbed polyharmonic Dirichlet boundary value problem by pointwise estimates.* PhD thesis, Otto-von-Guericke-Universität Magdeburg, 2015. DOI: 10.25673/4208.

[PW84] Murray H. Protter and Hans F. Weinberger. Maximum principles in differential equations. Corr. reprint. New York etc.: Springer-Verlag. X, 261 p. DM 79.00 (1984)., 1984.

[Rot94] Walter Roth. A combined approach to the fundamental theorems for normed spaces. *Bull. Inst. Math., Acad. Sin.*, 22(1):83–89, 1994.

[RS80] Michael Reed and Barry Simon. Methods of modern mathematical physics. I: Functional analysis. Rev. and enl. ed. New York etc.: Academic Press, A Subsidiary of Harcourt Brace Jovanovich, Publishers, XV, 400 p. $ 24.00 (1980)., 1980.

[Rud91] Walter Rudin. *Functional analysis.* International series in pure and applied mathematics. New York, NY: McGraw-Hill, 1991.

[SA17] Fatemeh Shakeri and Rahim Alizadeh. Nonnegative and eventually positive matrices. *Linear Algebra Appl.*, 519:19–26, 2017.

[Sch60] Helmut H. Schaefer. Some spectral properties of positive linear operators. *Pac. J. Math.*, 10:1009–1019, 1960.

[Sch74] Helmut H. Schaefer. *Banach lattices and positive operators.* Die Grundlehren der mathematischen Wissenschaften, Band 215. Springer-Verlag, New York-Heidelberg, 1974.

[Sch21] René L. Schilling. *Brownian motion. A guide to random processes and stochastic calculus. With a chapter on simulation by Björn Böttcher.* De Gruyter Grad. Berlin: De Gruyter, 3rd revised and extended edition edition, 2021.

[Sen06] Eugene Seneta. *Non-negative matrices and Markov chains.* Springer Ser. Stat. New York, NY: Springer, revised reprint of the 2nd ed. edition, 2006.

[SG01] Guido Sweers and Hans-Christoph Grunau. Optimal conditions for anti-maximum principles. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.*, 30(3-4):499–513, 2001.

[She51] Seymour Sherman. Order in operator algebras. *Am. J. Math.*, 73:227–232, 1951.

[Soo19] Aivar Sootla. Properties of eventually positive linear input-output systems. *IET Control Theory Appl.*, 13(7):891–897, 2019.

[SS20] Inka Schnieders and Guido Sweers. A biharmonic converse to Krein-Rutman: a maximum principle near a positive eigenfunction. *Positivity*, 24(3):677–710, 2020.

[Ste70] Elias M. Stein. *Singular integrals and differentiability properties of functions*, volume 30 of *Princeton Math. Ser.* Princeton University Press, Princeton, NJ, 1970.

[Sto32] M. H. Stone. On one-parameter unitary groups in Hilbert space. *Ann. Math. (2)*, 33:643–648, 1932.

[Str03] Robert S. Strichartz. *A guide to distribution theory and Fourier transforms.* River Edge, NJ: World Scientific, 2003.

[Swe16] Guido Sweers. On sign preservation for clotheslines, curtain rods, elastic membranes and thin plates. *Jahresber. Dtsch. Math.-Ver.*, 118(4):275–320, 2016.

[Tak96] Peter Takáč. An abstract form of maximum and anti-maximum principles of Hopf's type. *J. Math. Anal. Appl.*, 201(2):339–364, 1996.

[TCDF15] Francesco Tudisco, Valerio Cardinali, and Carmine Di Fiore. On complex power nonnegative matrices. *Linear Algebra Appl.*, 471:449–468, 2015.

[TRH01] Pablo Tarazaga, Marcos Raydan, and Ana Hurman. Perron-Frobenius theorem for matrices with some negative entries. *Linear Algebra Appl.*, 328(1-3):57–68, 2001.

[vN97] Jan M. A. M. van Neerven. The norm of a complex Banach lattice. *Positivity*, 1(4):381–390, 1997.

[Vog22] Hendrik Vogt. Stability of uniformly eventually positive C_0-semigroups on L_p-spaces. *Proc. Am. Math. Soc.*, 150(8):3513–3515, 2022.

[Voi88] Jürgen Voigt. The projection into the center of operators in a Banach lattice. *Math. Z.*, 199(1):115–117, 1988.

[Wei95] Lutz Weis. The stability of positive semigroups on L_p spaces. *Proc. Am. Math. Soc.*, 123(10):3089–3094, 1995.

[Wei98] Lutz Weis. A short proof for the stability theorem for positive semigroups on L_p. *Proc. Am. Math. Soc.*, 126(11):3253–3256, 1998.

[Wnu99] Witold Wnuk. *Banach lattices with order continuous norms.* Warsaw: Polish Scientific Publishers PWN, 1999.

[Wul17] Boris Zacharowitsch Wulich. *Geometrie der Kegel: in normierten Räumen.* De Gruyter Stud. Berlin: De Gruyter, 2017.

[Yos48] Kôsaku Yosida. On the differentiability and the representation of one-parameter semi-group of linear operators. *J. Math. Soc. Japan*, 1:15–21, 1948.

[Yos95] K. Yosida. *Functional analysis.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the sixth (1980) edition.

[Zaa83] Adriaan C. Zaanen. *Riesz spaces II*, volume 30 of *North-Holland Math. Libr.* Elsevier (North-Holland), Amsterdam, 1983.

[Zaa97] Adriaan C. Zaanen. *Introduction to operator theory in Riesz spaces.* Springer-Verlag, Berlin, 1997.

[ZT99] Boris G. Zaslavsky and Bit-Shun Tam. On the Jordan form of an irreducible matrix with eventually non-negative powers. *Linear Algebra Appl.*, 302/303:303–330, 1999. Special issue dedicated to Hans Schneider (Madison, WI, 1998).