Chapter 11

Eventually positive semigroups

In the previous chapter, we generalised the concept of matrix semigroups to infinite dimensions and had a glimpse into the world of positive semigroups. From now on, we focus all our attention to eventually positive C_0-semigroups. The present chapter is devoted to introducing the notion and spectral properties.

11.1 Individual and uniform eventual positivity

The following notions are at the heart of the rest of the course:

Definition 11.1.1 (Eventual positivity of resolvents)**.** Let $(e^{tA})_{t\geq 0}$ be C_0-semigroup on a Banach lattice E. Let $u \in E_+$ and $0 \leq Q \in \mathcal{L}(E)$.

(a) $(e^{tA})_{t\geq 0}$ is called **individually eventually positive with respect to** u if for each $0 \lneq x \in E$ there exists $t_0 \geq 0$ such that $e^{tA}x \geq u$ for all $t \geq t_0$.

(b) $(e^{tA})_{t\geq 0}$ is called **uniformly eventually positive with respect to** Q if there exists $t_0 \geq 0$ such that $e^{tA} \geq Q$ for all $t \geq t_0$.

The very attentive reader may recall from Exercise 2.2 that for matrix semigroups (i.e. C_0-semigroups on finite-dimensional Banach lattices, if you want to sound fancy) and the choices $u = 0$ and $Q = 0$, the individual and uniform notions of eventual positivity are equivalent. In infinite dimensions, the concepts are indeed distinct:

Example 11.1.2 (Individual versus uniform eventual positivity)**.** There exists an operator $A \in \mathcal{L}\big(\mathrm{C}([-1,1]\big)$ with the following properties:

(a) $(e^{tA})_{t\geq 0}$ is individually eventually positive with respect to 0.

(b) $(e^{tA})_{t\geq 0}$ is not uniformly eventually positive with respect to 0.

Proof. We consider the operator from Example 6.1.2, i.e. the block diagonal operator

$$A := \begin{pmatrix} 0 & 0 \\ 0 & -S-2 \end{pmatrix} \in \mathcal{L}\big(\mathbb{C}\,\mathbb{1} \oplus \ker\varphi\big).$$

Here $\varphi \in \mathrm{C}([-1,1])'$ is given by $\langle \varphi, f\rangle = \frac{1}{2}\int_{-1}^{1} f(\omega)\,\mathrm{d}\omega$ and S is the reflection operator on $\ker\varphi$ given by $Sf(\omega) = f(-\omega)$ for all $f \in \ker\varphi$ and $\omega \in [-1,1]$.

(a) By using $S^2 = \mathrm{id}_{\ker\varphi}$ one can readily check for $t \geq 0$,

$$e^{tA} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2t}\big(\cosh t - \sinh t\, S\big) \end{pmatrix}.$$

In particular, $e^{tA}g \to 0$ as $t \to \infty$ for all $g \in \ker\varphi$. Now let $0 \leq f \in \mathrm{C}([-1,1])$ and write f as $f = \langle \varphi, f\rangle \mathbb{1} + g$ for a function $g \in \ker\varphi$. Of course, $e^{tA}f = \langle \varphi, f\rangle \mathbb{1} + e^{tA}g \to \langle \varphi, f\rangle \mathbb{1}$ with respect to $\|\cdot\|_\infty$ for $t \to \infty$. Since $e^{tA}f$ is real and $\langle \varphi, f\rangle \mathbb{1}$ is a constant function with value $\langle \varphi, f\rangle > 0$, it follows that $e^{tA}f \geq 0$ for all sufficiently large $t \geq t_0$.

(b) Fix a number $t > 0$. We show that $e^{tA} \not\geq 0$. For each $\varepsilon > 0$, one can choose $f_\varepsilon \in \mathrm{C}([-1,1])_+$ that satisfies $f_\varepsilon(1) = 1$, $f_\varepsilon(-1) = 0$ and $\langle \varphi, f_\varepsilon\rangle = \varepsilon$. Again, we write f_ε as $f_\varepsilon = \varepsilon \mathbb{1} + g_\varepsilon$ for $g_\varepsilon \in \ker\varphi$. Since $g_\varepsilon(-1) = -\varepsilon$ and $g_\varepsilon(1) = 1 - \varepsilon$, we get

$$e^{tA}f_\varepsilon(-1) = \varepsilon + e^{-2t}\big(\cosh t\, g_\varepsilon(-1) - \sinh t\, g_\varepsilon(1)\big) \xrightarrow{\varepsilon \downarrow 0} -e^{-2t}\sinh t < 0.$$

So there exists $\varepsilon > 0$ such that $e^{tA}f_\varepsilon(-1) < 0$. Thus $e^{tA} \not\geq 0$ because $f_\varepsilon \geq 0$. ☐

Our goal in this chapter is to prove several spectral results for eventually positive semigroups. We do this in Section 11.4. As preparation we first discuss the relation between the spectral bound and the growth bound for C_0-semigroups (Section 11.2) and a few properties of Laplace transforms of vector-valued functions (Section 11.3).

11.2 Spectral bound vs. the growth bound

Let $A\colon X \supseteq \mathrm{dom}(A) \to X$ be the generator of a C_0-semigroup on a Banach space X. We recall the definition of the **growth bound** $\omega_0(A) \in [-\infty,\infty)$ of A from Definition 10.1.6. In terms of the semigroup, it can be written as

$$\omega_0(A) = \inf\big\{\omega \in \mathbb{R} : \exists\, M \geq 1 \text{ with } \big\|e^{tA}\big\| \leq Me^{\omega t}\ \forall\ t \geq 0\big\}.$$

On the other hand, if the scalar field is complex, then the **spectral bound** $\mathrm{s}(A) \in [-\infty,\infty)$ is, according to Definition 3.3.3, given by

$$\mathrm{s}(A) := \sup\big\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\big\}.$$

In this section, we analyse the relation between the spectral and growth bounds of semigroup generators. The results in this section are used in the next section and the subsequent chapters to better understand eventually positive semigroups. In the finite-dimensional case one can easily deduce from Proposition 1.3.6 that $\mathrm{s}(A) = \omega_0(A)$. However, for general C_0-semigroups in infinite-dimensions, the situation is more involved. Let us first note that at least the inequality $\mathrm{s}(A) \leq \omega_0(A)$ is always true.

Proposition 11.2.1. *Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a complex Banach space X.*

(a) *For* $\lambda \in \mathbb{C}$*, if the limit*

$$R_\lambda x := \lim_{t\to\infty} \int_0^t e^{-s\lambda} e^{sA} x \, \mathrm{d}s$$

exists for each $x \in X$*, then* $R_\lambda \in \mathcal{L}(X)$*,* $\lambda \in \rho(A)$*, and* $\mathcal{R}(\lambda, A) = R_\lambda$*.*

(b) *One has* $\mathrm{s}(A) \le \omega_0(A)$*.*

Proof. (a) The proof resembles arguments already used in Chapter 10. Proving the proposition is thus a good opportunity to practise those arguments, so we outsource the proof to Exercise 11.1.

(b) Suppose $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0(A)$, and choose $\omega \in (\omega_0(A), \operatorname{Re}\lambda)$. Then there exists $M \ge 1$ such that $\|e^{tA}\| \le Me^{\omega t}$ for all $t \ge 0$. Hence, for every $x \in X$, the map

$$[0,\infty) \to X, \qquad t \mapsto e^{-t\lambda} e^{tA} x$$

is Bochner integrable, so $\lambda \in \rho(A)$ by (a). □

One can construct an example of a (positive) C_0-semigroup – even of a positive C_0-semigroup on a Banach lattice – where one actually has $\mathrm{s}(A) < \omega_0(A)$. Such an example is discussed in Exercise 11.2. Hence, knowing the spectrum of the generator of a C_0-semigroup does not suffice in general to conclude whether it converges exponentially to 0. Note that this is quite in contrast to finite-dimensions (Proposition 1.3.6). However, there are other useful conditions to ensure exponential convergence to zero, which we discuss now. First, we observe that convergence to 0 in operator norm is the same as exponential convergence to 0 (in operator norm).

Proposition 11.2.2. *For a* C_0*-semigroup* $(e^{tA})_{t\ge 0}$ *on a Banach space, the following are equivalent:*

(i) $\omega_0(A) < 0$*, i.e. there exist* $M \ge 0$ *and* $\omega < 0$ *such that* $\|e^{tA}\| \le Me^{\omega t}$ *for all* $t \ge 0$*.*

(ii) $\|e^{tA}\| \to 0$ *as* $t \to \infty$*.*

(iii) *There exists a* $t_0 > 0$ *such that* $\|e^{t_0 A}\| < 1$*.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) are obvious.

"(iii) ⇒ (i)": Without loss of generality, we assume $\|e^{t_0 A}\| > 0$. Let $t > 0$ and $n \in \mathbb{N}$ be such that $t \in [nt_0, (n+1)t_0)$. Then for constants $M := \|e^{t_0 A}\|^{-1} \sup_{s\in[0,t_0]} \|e^{sA}\|$ and $\omega := t_0^{-1} \log \|e^{t_0 A}\| < 0$, we obtain for each $t > 0$ that

$$\|e^{tA}\| = \left\|\left(e^{t_0 A}\right)^n e^{(t-nt_0)A}\right\| \le \sup_{s\in[0,t_0]} \|e^{sA}\| \left\|\left(e^{t_0 A}\right)^n\right\| \le M \left\|\left(e^{t_0 A}\right)^{t/t_0}\right\| = Me^{\omega t}. \qquad \square$$

Before we give the next condition for $\omega_0(A) < 0$, let us briefly discuss the situation in the discrete-time case, i.e. for powers of bounded operators. For an operator $T \in \mathcal{L}(X)$ on a complex Banach space X, recall that the number

$$\mathrm{r}(T) := \sup\left\{|\lambda| : \lambda \in \sigma(T)\right\} \in [0,\infty)$$

is called the **spectral radius** of T. If $X \neq \{0\}$, the supremum is actually a maximum since the spectrum is a non-empty compact set. It is a classical result in the spectral theory of bounded linear operators [Yos95, Theorems VIII.2.3 and VIII.2.4] that

$$\mathrm{r}(T) = \inf_{k \in \mathbb{N}_0} \left\| T^k \right\|^{1/k} = \lim_{k \to \infty} \left\| T^k \right\|^{1/k}. \tag{11.2.1}$$

From this formula, one gets a similar characterisation of $\|T^n\| \to 0$ as in the finite dimensional case (Proposition 1.2.2):

Proposition 11.2.3 (Convergence to 0 of operator powers)**.** *For an operator $T \in \mathcal{L}(X)$ on a complex Banach space X, the following are equivalent:*

(i) $\mathrm{r}(T) < 1$.

(ii) *$T^k \to 0$ as $k \to \infty$.*

(iii) *There exist numbers $\eta \in [0,1)$ and $c \geq 0$ such that $\left\|T^k\right\| \leq c\eta^k$ for each $k \in \mathbb{N}_0$.*

(iv) *For some/all $p \in [1,\infty)$ one has*

$$\sum_{k=0}^{\infty} \left\| T^k x \right\|^p < \infty \qquad \forall\, x \in X.$$

Proof. "(ii) ⇔ (iii)": The proof of this equivalence is similar to the continuous-time case (Proposition 11.2.2), so we omit the details.

"(i) ⇒ (iii)": This follows from the spectral radius formula (11.2.1).

"(iii) ⇒ (iv) *for all p*": This implication is obvious.

"(iv) *for some p* ⇒ (i)": Assume that (iv) holds for some $p \in [1,\infty)$. Then the mapping $X \to \ell^p(\mathbb{N}_0; X)$, $x \mapsto (T^k x)_{k \in \mathbb{N}_0}$ is linear and closed, thus continuous by the closed graph theorem. Hence, there exists a number $M \geq 0$ such that

$$\sum_{k=0}^{\infty} \left\| T^k x \right\|^p \leq M^p \|x\|^p \quad \forall\, x \in X.$$

Choose a $\lambda \in \sigma(T)$ with $|\lambda| = \mathrm{r}(T)$. Since $\lambda \in \partial\sigma(T)$, it is an approximate eigenvalue of T, i.e. there is a sequence (x_n) in X such that $\|x_n\| = 1$ for all n and $(\lambda - T)x_n \to 0$ as $n \to \infty$ (Exercise 5.2). For every $k \in \mathbb{N}$, it follows that $(\lambda^k - T^k)x_n \to 0$ as $n \to \infty$

because the operator $(\lambda^k - T^k)$ is a multiple of $\lambda - T$. In particular, $\left\| T^k x_n \right\| \to |\lambda|^k$ as $n \to \infty$ for each $k \in \mathbb{N}_0$. Hence, by Fatou's lemma

$$M^p \geq \liminf_{n\to\infty} \sum_{k=0}^{\infty} \left\| T^k x_n \right\|^p \geq \sum_{k=0}^{\infty} |\lambda|^{kp}.$$

Consequently, $1 > |\lambda| = \mathrm{r}(T)$. □

Comparing Proposition 11.2.1(b) and Proposition 11.2.3 shows the difference between the powers T^k of bounded operators $T \in \mathcal{L}(X)$ and a C_0-semigroup $(e^{tA})_{t\geq 0}$: while the long-term behaviour of $\left\| T^k \right\|$ can be characterised in terms of $\mathrm{r}(T)$, the long-term behaviour of $\left\| e^{tA} \right\|$ is, in general, not completely characterised by $\mathrm{s}(A)$ (Exercise 11.2).

Motivated by Proposition 11.2.3(iv), it is natural to consider p-integrability of the orbits of a C_0-semigroup. Observe that

$$\omega_0(A) < 0 \quad \Rightarrow \quad p\text{-integrability of the orbits} \quad \Rightarrow \quad \mathrm{s}(A) < 0,$$

where the first implication is obvious and the second implication follows from Proposition 11.2.1(a). We now show that the first implication is in fact an equivalence.

Theorem 11.2.4 (Datko–Pazy). *Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a Banach space X and let $p \in [1,\infty)$. If*

$$\int_0^\infty \left\| e^{tA} x \right\|^p \, \mathrm{d}t < \infty \quad \forall\, x \in X,$$

then $\omega_0(A) < 0$.

Proof. A similar argument as in the proof of Proposition 11.2.3 shows that the estimate $\int_0^\infty \left\| e^{tA} x \right\|^p \, \mathrm{d}t < \infty$ is automatically uniform – i.e. there exists a number $M \geq 0$ such that

$$\int_0^\infty \left\| e^{tA} x \right\|^p \, \mathrm{d}t \leq M^p \left\| x \right\|^p \quad \forall\, x \in X.$$

Instead of the sequence space $\ell^p(\mathbb{N}_0; X)$, one has to work with the space $L^p([0,\infty); X)$ of X-valued p-integrable functions which is, for instance, explained in [HvNVW16, Section 1.2.b] and which we have not discussed. Thus we omit the details and trust that readers not familiar with Bochner spaces will find the analogy with the same argument in Proposition 11.2.3 convincing.

We proceed to prove that $\omega_0(A) < 0$. According to Proposition 11.2.2, it suffices to show that there exists a $k \in \mathbb{N}$ such that $\left\| e^{kA} \right\| < 1$. For this, it suffices by Proposition 11.2.3 to prove that $\mathrm{r}(e^A) < 1$. Hence choose $\lambda \in \sigma(e^A)$ with $|\lambda| = \mathrm{r}(e^A)$. Now we use a similar argument as in the proof of Proposition 11.2.3: since $\lambda \in \partial\sigma(e^A)$, it is an approximate eigenvalue of e^A (Exercise 5.2), i.e. there exists a sequence (x_n) in X such that $\|x_n\| = 1$ for each n and $(\lambda - e^A)x_n \to 0$ as $n \to \infty$. For each integer $k \in \mathbb{N}_0$ it follows that $(\lambda^k - e^{kA})x_n \to 0$ as $n \to \infty$.

Set $C := \sup_{t\in[0,1]} \left\|e^{tA}\right\| \in [1,\infty)$. For each $k \in \mathbb{N}_0$ and each $t \in [0,1]$ one can use $e^{(1-t)A}e^{tA} = e^A$ to obtain $C\left\|e^{tA}e^{kA}x_n\right\| \ge \left\|e^{(k+1)A}x_n\right\| \to |\lambda|^{k+1}$ as $n \to \infty$. Moreover, for each n, one has

$$M^p \ge \int_0^\infty \left\|e^{tA}x_n\right\|^p \, \mathrm{d}t = \sum_{k=0}^\infty \int_0^1 \left\|e^{tA}e^{kA}x_n\right\|^p \, \mathrm{d}t,$$

so Fatou's lemma gives $M^p \ge \sum_{k=0}^\infty |\lambda|^{(k+1)p}/C^p$. This shows that $1 > |\lambda| = \mathrm{r}(T)$. ☐

11.3 Intermezzo: The Laplace transform of positive vector-valued functions

Let $(e^{tA})_{t\ge 0}$ be a C_0-semigroup on a complex Banach space X. For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0(A)$, the integral $\int_0^\infty e^{-t\lambda}e^{tA}x \, \mathrm{d}t$ exists for $x \in X$ as a Bochner integral and is, according to Proposition 11.2.1(a), equal to $\mathcal{R}(\lambda, A)x$. As discussed in the previous section and demonstrated in Exercise 11.2, it can happen that $\mathrm{s}(A) < \omega_0(A)$. To infer information about the spectral bound $\mathrm{s}(A)$ from information about the semigroup, it is quite useful to have a similar representation of $\mathcal{R}(\lambda, A)x$ even for $\lambda > \mathrm{s}(A)$. In some cases, this is indeed possible. It follows from Theorem 11.2.4 for $p = 1$ that the integral $\int_0^\infty e^{-t\lambda}e^{tA}x \, \mathrm{d}t$ cannot exist as a Bochner integral for each $x \in X$ if $\operatorname{Re}\lambda \in (\mathrm{s}(A), \omega_0(A)]$. However, it does sometimes exists as an **improper integral**, i.e. as the limit $\lim_{\tau\to\infty}\int_0^\tau e^{-\lambda t}e^{tA}x \, \mathrm{d}t$. In particular, this is always the case for eventually positive C_0-semigroups, as we show in Theorem 11.4.1. For the proof of this theorem, we need a general result about the Laplace transform of positive vector-valued functions (Theorem 11.3.2).

To this end, we need, in turn, another result first. Recall that Dini's theorem says that, for a compact metric (or topological Hausdorff) space K and an increasing sequence (f_n) in $\mathrm{C}(K;\mathbb{R})$ that converges pointwise to a function $f \in \mathrm{C}(K;\mathbb{R})$, the convergence is automatically uniform. A similar result can be shown on Banach lattices: there, weak convergence of an increasing sequence implies norm convergence. This is the content of the following theorem (which we formulate even for nets, making the result a bit more convenient to use). For increasing sequences in $\mathrm{C}(K;\mathbb{R})$, Dini's classical theorem can be obtained from the monotone convergence theorem together with the Riesz representation theorem of the dual space $\mathrm{C}(K;\mathbb{R})$.

Theorem 11.3.1 (Dini's theorem in Banach lattices)**.** *Let E be a Banach lattice and let (x_j) be an increasing net of real elements of E, i.e. $x_{j_1} \le x_{j_2}$ for $j_1 \le j_2$. If (x_j) converges weakly to an element $x \in E$, then it even converges in norm to x.*

Proof. First note that $x \ge x_j$, since inequalities can be checked by testing against positive functionals (Proposition 4.4.4). Let B'_+ denote the positive closed unit ball of E'. Then

$$\left\|x - x_j\right\| = \sup_{\|z'\|\le 1} \left|\left\langle z', x - x_j\right\rangle\right| \le \sup_{\|z'\|\le 1} \left\langle \left|z'\right|, x - x_j\right\rangle = \sup_{x'\in B'_+} \left\langle x', x - x_j\right\rangle,$$

where the inequality follows from the inequality at the end of Theorem 4.4.2. Therefore, it suffices to show that

$$\sup_{x' \in B'_+} \langle x', x - x_j \rangle \to 0. \tag{11.3.1}$$

To this end, let $\varepsilon > 0$ and consider the sets

$$B'_j := \{x' \in B'_+ : \langle x', x - x_j \rangle < \varepsilon\}.$$

Each of these sets is open in B'_+ with respect to the weak* topology. Moreover, since $x_j \to x$ weakly, one has $B'_+ = \bigcup_j B'_j$. As B'_+ is weak* compact by the Banach–Alaoglu theorem, there exist finitely many indices $j_1, \dots, j_\ell$ such that $B'_+ = B'_{j_1} \cup \dots \cup B'_{j_\ell}$. Choose an index j_0 that dominates $j_1, \dots, j_\ell$. Since the net (x_j) is increasing it follows that $\langle x', x - x_j \rangle < \varepsilon$ for all $x' \in B'_+$ and all $j \geq j_0$, which proves (11.3.1). □

By means of Dini's theorem we can now show the theorem on Laplace transforms of positive vector-valued functions that we promised at the beginning of the section. For every $\gamma \in [-\infty, \infty)$, we define the open right-half plane $\mathbb{C}_{\mathrm{Re}>\gamma} := \{z \in \mathbb{C} : \mathrm{Re}\, z > \gamma\}$.

Theorem 11.3.2. *Let $-\infty \leq \gamma < \omega < \infty$. Let E be a complex Banach space and let $f : [0,\infty) \to E$ be a continuous function that satisfies $\|f(t)\| \leq Me^{\omega t}$ for a number $M \geq 0$ and all $t \geq 0$. Define $\hat{f} : \mathbb{C}_{\mathrm{Re}>\omega} \to E$ by*

$$\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t) \,\mathrm{d}t \quad \forall\, \lambda \in \mathbb{C}_{\mathrm{Re}>\omega}.$$

(a) *The function $\hat{f}$ is analytic.*

(b) *Assume now that E is a complex Banach lattice, that $f(t) \geq 0$ for all $t \geq 0$, and that $\hat{f}$ extends to a analytic function from $\mathbb{C}_{\mathrm{Re}>\gamma}$ to E, which we again denote by $\hat{f}$. Then*

$$\hat{f}(\lambda) = \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} f(t) \,\mathrm{d}t \quad \forall\, \lambda \in \mathbb{C}_{\mathrm{Re}>\gamma}.$$

Proof. (a) One can prove that the dominated convergence theorem holds for sequences of Bochner integrable functions which are dominated in norm by an integrable scalar-valued function [ABHN11, Theorem 1.1.8]. This implies that $\hat{f}$ is analytic, and its derivatives are given by the formula

$$\hat{f}^{(k)}(\mu) = \int_0^\infty (-t)^k e^{-\mu t} f(t) \,\mathrm{d}t \tag{11.3.2}$$

for all $k \in \mathbb{N}_0$ and all $\mu \in \mathbb{C}_{\mathrm{Re}>\omega}$, which we use in the proof of (b).

(b) The result is clear for $\lambda \in \mathbb{C}_{\mathrm{Re}>\omega}$.

Next we consider real numbers $\lambda \in (\gamma, \omega]$, so fix such a λ. Choose an arbitrary number $\mu > \omega$. Then there is an open disk in $\mathbb{C}_{\mathrm{Re}>\gamma}$ with centre μ that contains λ, so the Taylor series for analytic functions gives

$$\hat{f}(\lambda) = \sum_{k=0}^{\infty} \frac{(\mu-\lambda)^k}{k!} (-1)^k \hat{f}^{(k)}(\mu)$$

Consider a positive functional $0 \le x' \in E'$. By testing the previous equality against x' and using formula (11.3.2) we obtain

$$\begin{aligned} \langle x', \hat{f}(\lambda) \rangle &= \sum_{k=0}^{\infty} \frac{(\mu-\lambda)^k}{k!} \int_0^\infty t^k e^{-\mu t} \langle x', f(t) \rangle \, \mathrm{d}t \\ &= \int_0^\infty \langle x', f(t) \rangle e^{-\mu t} \sum_{k=0}^{\infty} \frac{(\mu-\lambda)^k t^k}{k!} \, \mathrm{d}t \\ &= \int_0^\infty \langle x', f(t) \rangle e^{-\lambda t} \, \mathrm{d}t = \lim_{\tau\to\infty} \left\langle x', \int_0^\tau e^{-\lambda t} f(t) \, \mathrm{d}t \right\rangle, \end{aligned}$$

where the positivity was used to swap the series and the integral. Since the dual space E' is a complex Banach lattice, it is spanned by the positive functionals. Hence, it follows that the increasing net $\left(\int_0^\tau e^{-\lambda t} f(t) \, \mathrm{d}t\right)_{\tau\in(0,\infty)}$ converges weakly to $\hat{f}(\lambda)$. According to the version of Dini's theorem for Banach lattices (Theorem 11.3.1) the convergence even takes place in norm.

It remains to show that the result holds not only on the real line, but also for all λ with $\mathrm{Re}\,\lambda > \gamma$. This is a general property of vector-valued Laplace transforms that is not related to positivity, so we only give a reference: the interested reader can find the details in [ABHN11, Proposition 1.4.1 and Theorem 1.5.1]. ☐

11.4 The spectrum and the resolvent of an eventually positive semigroup

By Theorem 11.3.2(b), we immediately obtain that for positive C_0-semigroups, the Laplace transform representation of the resolvent (Theorem 10.2.3(b)) extends to the right half-plane $\mathbb{C}_{\mathrm{Re}>\mathrm{s}(A)}$. In fact, this is also the case for eventual positivity:

Theorem 11.4.1. *On a complex Banach lattice E, let $(e^{tA})_{t\ge0}$ be a C_0-semigroup that is individually eventually positive with respect to* 0. *Then*

$$\mathcal{R}(\lambda, A)x = \lim_{\tau\to\infty} \int_0^\tau e^{-\lambda s} e^{sA} x \, \mathrm{d}s \quad \forall\, x \in X$$

for all $x \in X$ and all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > \mathrm{s}(A)$.

Proof. We may assume $x \ge 0$. Choose $t_0 \ge 0$ such that $e^{tA} \ge 0$ for all $t \ge t_0$ and define $f : [0,\infty) \to E$ as $f(t) = e^{(t+t_0)A} x$ for all $t \ge 0$. By Theorem 10.2.3(b),

$$\mathcal{R}(\lambda, A)x = \int_0^\infty e^{-\lambda s} e^{sA} x \, \mathrm{d}s = \int_0^{t_0} e^{-\lambda s} e^{sA} x \, \mathrm{d}s + e^{-\lambda t_0} \hat{f}(\lambda)$$

for all $\lambda \in \mathbb{C}_{\mathrm{Re}>\omega_0(A)}$. Since f is positive and $\hat{f}$ extends to an analytic function from $\mathbb{C}_{\mathrm{Re}>\mathrm{s}(A)}$ to E, the assertion follows from Theorem 11.3.2(b). ☐

By means of the previous theorem we can now show the following to results about the spectrum of the generator of eventually positive semigroups.

Theorem 11.4.2 (The spectral bound of eventually positive semigroups). *Let $(e^{tA})_{t\geq 0}$ be C_0-semigroup on a complex Banach lattice E such that $\sigma(A) \neq \emptyset$. If $(e^{tA})_{t\geq 0}$ is individually eventually positive with respect to* 0*, then* $\mathrm{s}(A) \in \sigma(A)$.

Proof. By Proposition 3.3.2(a), $\mathrm{dist}(\lambda, \sigma(A)) \geq \|\mathcal{R}(\lambda, A)\|^{-1}$ for all $\lambda \in \rho(A)$. Therefore, there exists $(\lambda_n) \subseteq \rho(A)$ such that $\mathrm{Re}\,\lambda_n \downarrow \mathrm{s}(A)$ and $\|\mathcal{R}(\lambda_n, A)\| \to \infty$ as $n \to \infty$.

By the uniform boundedness principle, there exists $f \in E_{\mathbb{R}}$ and a subsequence (λ_{n_k}) of (λ_n) such that $\|\mathcal{R}(\lambda_{n_k}, A)f\| \to \infty$ as $k \to \infty$. On the other hand, Exercise 11.3 shows that there is a norm-bounded function $r_f\colon (\mathrm{s}(A), \infty) \to E$ such that

$$\left|\mathcal{R}(\lambda_{n_k}, A)f\right| \leq \mathcal{R}(\mathrm{Re}\,\lambda_{n_k}, A)\left|f\right| + r_f(\mathrm{Re}\,\lambda_{n_k}).$$

As $k \to \infty$, $r_f(\cdot)$ remains bounded in E, while norm of the term on the left tends to ∞. Hence $\|\mathcal{R}(\mathrm{Re}\,\lambda_{n_k}, A)\left|f\right|\| \to \infty$ as well. As $\mathrm{Re}\,\lambda_{n_k} \downarrow \mathrm{s}(A)$, we conclude $\mathrm{s}(A) \in \sigma(A)$. ☐

In Theorem 6.3.3 we showed spectral properties of an eigenvalue at which the resolvent of an operator is eventually positive. We now show an analogue of this result for eventually positive semigroups.

Theorem 11.4.3 (Eigenvectors for eventually positive semigroups). *On a complex Banach lattice E, let $(e^{tA})_{t\geq 0}$ be C_0-semigroup that is individually eventually positive with respect to* 0 *such that* $\mathrm{s}(A) > -\infty$. *Assume that* $\mathrm{s}(A)$ [1] *is a pole of* $\mathcal{R}(\cdot, A)$. *Then:*

(a) *Both A and A' have a positive eigenvector for the eigenvalue* $\mathrm{s}(A)$.

(b) *If* $\mathrm{s}(A)$ *is a semisimple eigenvalue, then its associated spectral projection is positive.*

Proof. We begin with a preliminary observation for eventually positive semigroups. Let $x \in E_+$ and $t_0 \geq 0$ such that $e^{tA}x \geq 0$ for all $t \geq t_0$. By Theorem 11.4.1, we can write $\mathcal{R}(\lambda, A)x = \int_0^{t_0} e^{-\lambda s} e^{sA} x \,\mathrm{d}s + y_\lambda$, where $y_\lambda = \lim_{\tau\to\infty} \int_{t_0}^{\tau} e^{-\lambda s} e^{sA} x \,\mathrm{d}s \geq 0$ for all $\lambda > \mathrm{s}(A)$. Therefore,

$$\begin{aligned}
\mathrm{dist}\big((\lambda - \mathrm{s}(A))\mathcal{R}(\lambda, A)x, E_+\big) &\leq \big\|(\lambda - \mathrm{s}(A))\mathcal{R}(\lambda, A)x - (\lambda - \mathrm{s}(A))y_\lambda\big\| \\
&= (\lambda - \mathrm{s}(A))\left\|\int_0^{t_0} e^{-\lambda s} e^{sA} x \,\mathrm{d}s\right\| \\
&\leq (\lambda - \mathrm{s}(A))(1 - e^{-\lambda t_0}) \sup_{s\in[0,t_0]} \|e^{sA}x\| \to 0
\end{aligned}$$

as $\lambda \downarrow \mathrm{s}(A)$; the last line uses the local boundedness of semigroup orbits (see Proposition 10.2.2).

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[1] We already know $\mathrm{s}(A) \in \sigma(A)$ from Theorem 11.4.2

(a) Let $p \in \mathbb{N}$ denote the pole order of $\mathrm{s}(A)$ and let Q_{-p+1} denote the coefficient of $(\lambda - \mathrm{s}(A))^{-p}$ in the Laurent series expansion of $\mathcal{R}(\cdot, A)$ about $\mathrm{s}(A)$. By Theorem 6.2.6, $\mathrm{s}(A)$ is an eigenvalue of A and A' with $\operatorname{rg} Q_{-p+1} \subseteq \ker(\mathrm{s}(A) - A)$ and $\operatorname{rg} Q'_{-p+1} \subseteq \ker(\mathrm{s}(A) - A')$. Moreover, $(\lambda - \mathrm{s}(A))^p \mathcal{R}(\lambda, A) \to Q_{-p+1}$ in operator norm as $\lambda \downarrow \mathrm{s}(A)$. It follows from the observation above that both Q_{-p+1} and Q'_{-p+1} are positive. Hence, both A and A' have a positive eigenvector for the eigenvalue $\mathrm{s}(A)$.

(b) If the eigenvalue λ is semisimple, then $p = 1$ and $0 \le Q_{-p+1} = Q_0$ is the spectral projection by Theorem 6.2.6(b) and (c). ☐

We observe that Theorems 11.4.3 and 11.4.2 together yield a Perron-Frobenius type result for eventually positive semigroups valid in infinite dimensions (cf. the finite dimensional case in Theorem 2.2.3).

Recall again that, for the generator A of a C_0-semigroup, one can sometimes have $\mathrm{s}(A) < \omega_0(A)$ even if the semigroup is positive (Exercise 11.2). This cannot happen if the underlying Banach lattice is sufficiently “nice”. We show this for two classes of Banach lattices in the following theorem.

Theorem 11.4.4. *On a complex Banach lattice E, let $(e^{tA})_{t\ge 0}$ be C_0-semigroup that is individually eventually positive with respect to* 0. *Then* $\mathrm{s}(A) = \omega_0(A)$ *in each of the following cases:*

(a) *E is an L^1-space or, more generally, the norm on E is additive on the positive cone.*[2]

(b) $E = \mathrm{C}_0(\Omega)$ *for a locally compact metric space* Ω.[3,4]

For the proof of (b), we use the following property of $C_0(\Omega)$.

Proposition 11.4.5. *Let Ω be a locally compact metric space.*[4] *If $\emptyset \ne F \subseteq \mathrm{C}_0(\Omega;\mathbb{R})$ is relatively compact, then F has a supremum and an infimum in the Banach lattice* $\mathrm{C}_0(\Omega)$.

Proof. It suffices to show that F has a supremum; the existence of the infimum then follows by considering $-F$.

Of course, F is bounded. Since one can identify $\mathrm{C}_0(\Omega)$ with $\{f \in \mathrm{C}(\Omega \cup \{\infty\}) : f(\infty) = 0\}$, where $\Omega \cup \{\infty\}$ denotes the one-point compactification of Ω, the Arzelà-Ascoli theorem applies to $\mathrm{C}_0(\Omega)$, and thus F is equicontinuous. Now consider the set

$$G := \{f_1 \vee \cdots \vee f_k \in \mathrm{C}_0(\Omega;\mathbb{R}) : k \in \mathbb{N},\ f_1, \dots, f_k \in F\}.$$

It is not difficult to check that G is also equicontinuous and bounded and in turn, relatively compact again by the Arzelà–Ascoli theorem. Moreover, G is a directed set with

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[2] Strictly speaking, the latter condition is not more general since there is a classical representation result saying that if the norm on a Banach lattice is additive on the cone, then the Banach lattice is isometrically lattice isomorphic to an L^1-space. In any case, additivity of the norm on E_+ is what is needed for the proof.

[3] **Local compactness** means that for every point $\omega \in \Omega$, every open neighbourhood U of ω contains a compact neighbourhood K of ω. For instance, every open subset of $\mathbb{R}^n$ is a locally compact space with respect to the Euclidean metric.

[4] Or, more generally, a locally compact topological Hausdorff space.

respect to the order induced from $C_0(\Omega;\mathbb{R})$, so $(g)_{g\in G}$ is an increasing and norm bounded net. Hence, it converges pointwise to a function $h\colon \Omega \to \mathbb{R}$. Using the equicontinuity of G, one can show that $h \in C_0(\Omega;\mathbb{R})$. Now it easily follows that h is the smallest upper bound of G in $C_0(\Omega;\mathbb{R})$. By the definition of G, this implies that h is also the smallest upper bound of F in $C_0(\Omega;\mathbb{R})$. ☐

Proof of Theorem 11.4.4. Due to Proposition 11.2.1(b), we only need to show that $\omega_0(A) \le \mathrm{s}(A)$. To this end, it is sufficient in each of the cases to show that the implication

$$\mathrm{s}(A) < 0 \quad\Rightarrow\quad \omega_0(A) \le 0$$

holds, because by applying this to the operator $A - \alpha$ for real numbers α one obtains

$$\mathrm{s}(A) < \alpha \quad\Rightarrow\quad \omega_0(A) \le \alpha,$$

and hence, $\omega_0(A) \le \mathrm{s}(A)$.

(a) We prove the result for the case that the semigroup is positive. The generalisation to the individually eventually positive case is not difficult for assertion (a) and is thus posed as Exercise 11.4. Assume that $\mathrm{s}(A) < 0$. We use the Datko–Pazy Theorem 11.2.4 for $p = 1$ to show $\omega_0(A) < 0$.

   Fix $0 \le x \in E$. By Theorem 11.4.1 and Proposition 11.2.1(a) it follows from $\mathrm{s}(A) < 0$ that $\int_0^\tau e^{tA}x\,\mathrm{d}t \to \mathcal{R}(0,A)x$ as $\tau \to \infty$. As the norm is additive on the positive cone, one can swap the norm with integral of positive E-valued functions, so we obtain

$$\int_0^\infty \left\| e^{tA} \right\| \mathrm{d}t = \lim_{\tau\to\infty} \int_0^\tau \left\| e^{tA} \right\| x \,\mathrm{d}t = \lim_{\tau\to\infty} \left\| \int_0^\tau e^{tA}\,\mathrm{d}t \right\| = \|\mathcal{R}(0,A)x\| < \infty.$$

   As E_+ spans E, we conclude that $\int_0^t \left\| e^{tA}x \right\| \mathrm{d}t < \infty$ even for all $x \in E$. Thus, Theorem 11.2.4 of Datko–Pazy gives $\omega_0(A) < 0$, as claimed.

(b) Let $\mathrm{s}(A) < 0$ and fix a vector $0 \le f \in E$. We will show that the semigroup orbit $(e^{tA}f)_{t\ge 0}$ is bounded. Since E_+ spans E, the uniform boundedness principle then implies that the semigroup is bounded, and thus $\omega_0(A) \le 0$.

   To make the main idea as transparent as possible, we first consider the case where the semigroup is positive. The individually eventually positive case is – in contrast to (a) – technically much more involved and is treated afterwards.

   For positive semigroups: Since $E = C_0(\Omega)$, Proposition 11.4.5 shows that there exists a function $g \in C_0(\Omega)_+$ such that $0 \le e^{tA}f \le g$ for all $t \in [0,1]$ (this is in fact the only part of the argument where we need that $E = C_0(\Omega)$).

   Since $\mathrm{s}(A) < 0$ and $e^{sA}g \ge 0$ for each $s \in [0,\infty)$, Theorem 11.4.1 and Proposition 11.2.1(a) imply that $\int_0^\tau e^{sA}g\,\mathrm{d}s \to \mathcal{R}(0,A)g$ as $\tau \to \infty$. Thus for every $t \ge 1$,

$$0 \le e^{tA}f = \int_0^1 e^{tA}f\,\mathrm{d}s = \int_0^1 e^{(t-s)A}\underbrace{e^{sA}f}_{\le g}\,\mathrm{d}s \le \int_0^1 e^{(t-s)A}g\,\mathrm{d}s$$

$$= \int_{t-1}^{t} e^{sA} g \, \mathrm{d}s \leq \lim_{\tau \to \infty} \int_0^{\tau} e^{sA} g \, \mathrm{d}s = \mathcal{R}(0, A) g;$$

the positivity of the semigroup was used for each of the inequalities. Hence the orbit $(e^{tA}f)_{t\geq 0}$ is order bounded, and in particular bounded.

For individually eventually positive semigroups: By the individual eventual positivity, there exists $t_f \geq 0$ such that $e^{tA}f \geq 0$ for all $t \geq t_f$. According to Proposition 11.4.5, there exists a $0 \leq g \in \mathrm{C}_0(\Omega)$ such that $e^{tA}f \leq g$ for all $t \in [t_f, t_f + 1]$. Again by the individual eventual positivity, we can find $t_g \geq 0$ such that $e^{tA}g \geq 0$ for all $t \geq t_g$.

Consider the non-empty compact set

$$P := \left\{ g - e^{tA} f : t \in [t_f, t_f + 1] \right\} \subseteq E_+.$$

For each $w \in P$, once again, due to the individual eventual positivity, we can find $n \in \mathbb{N}$ such that $e^{tA}w \geq 0$ for all $t \geq n$. As a result, $P = \bigcup_{n\in\mathbb{N}} P_n$, where

$$P_n := \left\{ w \in P : e^{tA} w \geq 0 \text{ for all } t \geq n \right\}.$$

By the Baire category theorem, we get that the interior $(P_k)^\circ$ of P_k within P is non-empty for some $k \in \mathbb{N}$.

Next, consider the function $\Phi : [t_f, t_f + 1] \to P$ given by $t \mapsto g - e^{tA}f$. Continuity and surjectivity of Φ jointly imply that the pre-image of $(P_k)^\circ$ under Φ is non-void and an open subset of $[t_f, t_f + 1]$, and so it contains an interval $[\alpha, \alpha + \beta]$ for some $\alpha \geq 0$ and $0 < \beta \leq 1$. Observe that the implications

$$t \in [\alpha, \alpha + \beta] \quad \Rightarrow \quad \Phi(t) \in (P_k)^\circ \quad \Rightarrow \quad e^{sA}\left(y - e^{tA} f\right) \geq 0$$

for all $s \geq \alpha' := \max\{k, t_g\}$. Lastly, we fix a time $t \geq \alpha + \alpha' + \beta$ and the interval $I := [t - \alpha - \beta, t - \alpha] \subseteq [\alpha', \infty)$. Then

$$s \in I \quad \Rightarrow \quad t - s \in [\alpha, \alpha + \beta] \quad \Rightarrow \quad e^{sA}\left(y - e^{(t-s)A} f\right) \geq 0.$$

We infer

$$0 \leq e^{tA} f = \frac{1}{\beta} \int_I e^{tA} f \, \mathrm{d}s \leq \frac{1}{\beta} \int_I e^{sA} g \, \mathrm{d}s \leq \frac{1}{\beta} \lim_{\tau \to \infty} \int_{\alpha'}^{\tau} e^{sA} g \, \mathrm{d}s =: \tilde{g} \in \mathrm{C}_0(\Omega)_+,$$

where the limit exists because $\mathrm{s}(A) < 0$ and the semigroup is individually eventually positive (Theorem 11.4.1). So $\|e^{tA}f\| \leq \|\tilde{g}\|$ for all $t \geq \tau + \tau' + \beta$ and hence, the orbit $(e^{tA}f)_{t\geq 0}$ is bounded. □

There are more spaces than L^1 and $\mathrm{C}_0(\Omega)$ where the quality $\mathrm{s}(A) = \omega_0(A)$ holds for (eventually) positive semigroups; see the Notes at the end of the chapter.

Exercises for Chapter 11

Exercise 11.1. Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a complex Banach space X and let $\lambda \in \mathbb{C}$. Assume that the limit

$$R_\lambda x := \lim_{t\to\infty} \int_0^t e^{-s\lambda} e^{sA} x \,\mathrm{d}s$$

exists for each $x \in X$.

(a) Show that $R_\lambda \colon X \to X$ is a bounded linear operator.

(b) Show that $\lambda \in \rho(A)$ and $\mathcal{R}(\lambda, A) = R_\lambda$.

Hint: Argue similarly as in the proof of (ii)⇒(i), Step 2, in Theorem 10.1.7.

Exercise 11.2. In this exercise we show that the spectral bound and the growth bound of a semigroup generator need not coincide, in general – even for a positive semigroup on a Banach lattice.

Consider the vector space $E := L^1\big([1,\infty)\big) \cap \mathrm{C}_0\big([1,\infty)\big)$, endowed with the norm given by $\|f\|_E := \|f\|_1 \vee \|f\|_\infty$ for all $f \in E$.

(a) Show that E is a complex Banach lattice with respect to the norm $\|\cdot\|_E$ and the pointwise order on the real-valued functions in E.

(b) Consider the family of positive operators $(T(t))_{t\geq 0}$ on E given by

$$T(t)f(x) = f(e^t x)$$

for all $f \in E$, $t \geq 0$, and $x \in [1,\infty)$. Show that $(T(t))_{t\geq 0}$ is a C_0-semigroup on E and that $\|T(t)\| = 1$ for each $t \geq 0$.

(c) Let $A \colon E \supseteq \operatorname{dom}(A) \to E$ denote the generator of the semigroup $(T(t))_{t\geq 0}$. Determine the growth bound $\omega_0(A)$.

(d) Show that $\mathrm{s}(A) \leq -1$. *Hint*: Use Exercise 11.1.

Exercise 11.3. Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a complex Banach lattice E that is individually eventually positive C_0-semigroup with respect to 0, and let $f \in E_\mathbb{R}$ be arbitrary.

(a) Show that there exists $t_0 \geq 0$ such that $\left|e^{tA} f\right| \leq e^{tA}\left|f\right|$ for all $t \geq t_0$. Why does this not follow from Proposition 4.3.2?

(b) Prove that there is a norm-bounded function $r_f \colon (\mathrm{s}(A), \infty) \to E$ such that

$$\left|\mathcal{R}(\lambda, A) f\right| \leq \mathcal{R}(\operatorname{Re}\lambda, A)\left|f\right| + r_f(\operatorname{Re}\lambda) \quad \forall\, \lambda \in \mathbb{C}_{\operatorname{Re}>\mathrm{s}(A)}. \tag{11.4.1}$$

Hint: for $\tau \geq t_0$, consider the modulus of the integral $\int_0^\tau e^{-\lambda t} e^{tA} f \, \mathrm{d}t$. Then use Theorem 11.4.1 to allow $\tau \to \infty$.

(c) Assume now that $\sigma(A) \neq \emptyset$ and A has compact resolvent. Theorem 11.4.2 and the spectral theory of compact operators (Theorem 6.2.9) then imply that $\mathrm{s}(A)$ is a pole of the resolvent, say of order $m \in \mathbb{N}$.

Show that every $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = \mathrm{s}(A)$ has pole order at most m.

Exercise 11.4. Adapt the proof of Theorem 11.4.4(a) to show that it remains true for C_0-semigroups which are individually eventually positive with respect to 0.

Notes for Chapter 11

Eventually positive semigroups

A systematic study of eventually positive semigroups in infinite dimensions was initiated in [DGK16b], which focused on $\mathrm{C}(K)$-spaces. The distinction of individual and uniform eventual positivity (Example 11.1.2) and the results in Section 11.4 are taken from this paper. The theory on general Banach lattices was then further developed in [DGK16a], which was the starting point for a large number of papers on eventual positivity over the past decade.

The Datko–Pazy theorem

Several different proofs for the Datko–Pazy Theorem 11.2.4 can be found in the literature. The proof in Pazy's book [Paz83, Theorem 4.1] is, in a sense, quite quantitative, while a more abstract argument can be found e.g. in [EN00, Theorem V.1.8]. The spectral theoretic proof that we gave is essentially taken from [ABHN11, Theorem 5.1.2], although we simplified the argument somewhat (and turned it from a proof by contradiction into a direct argument) by employing Fatou's lemma.

Equality of spectral bound and growth bound for positive semigroups

The equality $\mathrm{s}(A) = \omega_0(A)$ is sometimes known as **Lyapunov's theorem**. As we shall see in the next chapter, it always holds under suitable "regularity" assumptions on the semigroup. The equality $\mathrm{s}(A) = \omega_0(A)$ for positive semigroups on L^1-spaces is a classical result; its generalisation to eventually positive semigroups is immediate and can already be found in [DGK16b]. The same equality on $\mathrm{C}(K)$ for a compact space K is also not difficult to prove; in the eventually positive case for real A it is also given in [DGK16b].

The situation on $\mathrm{C}_0(\Omega)$ is more involved, since there exists no function in $\mathrm{C}_0(\Omega)_+$ that dominates a multiple of any other function (unless Ω is compact). For positive semigroups, classical proofs of this result rely on duality arguments: the dual space on $\mathrm{C}_0(\Omega)$ is a space of measures and its norm is additive on the positive cone, so it is natural to apply the L^1-result on the dual space. A major caveat with this approach is that duals of

C_0-semigroups on non-reflexive spaces need not be C_0-semigroups in general, so technical tools – so called *sun duals* – are needed to circumvent this problem. As those tools use the positivity for small times, they cannot be used in the eventually positive case. The proof that we gave for Theorem 11.4.4(b) is taken from [AG22b] and was inspired by Vogt's paper [Vog22]. We note in passing that the class of spaces can be slightly generalised from $\mathrm{C}_0(\Omega)$ to so-called **AM-spaces**, since Proposition 11.4.5 still holds on those spaces; but we refrain from discussing more details here.

There are further spaces where the equality $\mathrm{s}(A) = \omega_0(A)$ holds for (eventually) positive semigroups. On L^2-spaces this is a consequence of the Gearhart–Prüss theorem and the proof works just well for the individually eventually positive case as for the positive case [DGK16b]. On L^p-spaces for $p \in (1,\infty) \setminus \{2\}$, the equality $\mathrm{s}(A) = \omega_0(A)$ for positive semigroups is more difficult to see and was shown by Weis more than a decade after the same result was known on the other spaces [Wei95, Wei98]. A surprisingly simple proof was recently given by Vogt [Vog22] who also used his technique to generalise the result to uniformly eventually positive semigroups. For individually eventually semigroups the question of whether $\mathrm{s}(A) = \omega_0(A)$ always holds on L^p is still open for $p \in (1,\infty) \setminus \{2\}$.

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