Chapter 10

An introduction to (positive) C_0-semigroups

The previous chapters dealt mainly with positivity and eventual positivity of the resolvent $\mathcal{R}(\cdot, A)$ of a linear operator A, i.e. we discussed positivity of the solution u to the linear equation $(\lambda - A)u = f$ in case that f is positive. For the remaining part of the internet seminar, we come back to a different type of equation (already known from Section 1.3 in the finite-dimensional case): the differential equation $\dot{u}(t) = Au(t)$ for $t \geq 0$, where $A \colon X \supseteq \operatorname{dom}(A) \to X$ is a linear operator on a Banach space X. Typically, t is interpreted as the time variable, and thus we think of $\dot{u}(t) = Au(t)$ as an **evolution equation**, as it governs the time evolution of the unknown function $u \colon [0, \infty) \to X$.

10.1 Linear evolution equations

To study differential equations in Banach spaces we need derivatives of vector-valued functions. For an interval $I \subseteq \mathbb{R}$ and a Banach space X, differentiability and the derivative of $f \colon I \to X$ is defined in terms of differential quotients in the obvious ways. Recall that every continuous $f \colon I \to X$ is Bochner integrable on every compact interval $J \subseteq I$ (Example 4.A.9). For such f and $t_0, t \in I$, we use the common notation

$$\int_{t_0}^{t} f(s)\, \mathrm{d}s := \begin{cases} \int_{[t_0,t]} f \, \mathrm{d}\lambda_1 & \text{if } t \geq t_0, \\ -\int_{[t,t_0]} f \, \mathrm{d}\lambda_1 & \text{if } t < t_0; \end{cases}$$

where λ_1 is the one-dimensional Lebesgue measure. As in the scalar-valued case, both parts of the fundamental theorem of calculus hold, i.e. $\frac{\mathrm{d}}{\mathrm{d}t} \int_{t_0}^{t} f(s)\, \mathrm{d}s = f(t)$ if $f \in \mathrm{C}(I; X)$ and $\int_{t_0}^{t} f'(s)\, \mathrm{d}s = f(t) - f(t_0)$ if $f \in \mathrm{C}^1(I; X)$.

Let us now specify the differential equation that we study throughout the rest of the course, along with two different solution concepts.

Definition 10.1.1 (Abstract Cauchy problems and their solutions)**.** Let X be a Banach space and let $A \colon X \supseteq \operatorname{dom}(A) \to X$ be a linear operator.

(a) For a function $u\colon [0,\infty) \to X$ (whose regularity is to be specified below) and a vector $x \in X$, the initial value problem

$$\begin{cases} \dot{u}(t) = Au(t), & t \in [0,\infty), \\ u(0) = x \end{cases} \tag{ACP}$$

is called an **abstract Cauchy problem**.

(b) Let $x \in X$. A function $u\colon [0,\infty) \to X$ is called a **classical solution**[1] of (ACP) if $u \in \mathrm{C}^1([0,\infty); X)$, $u(t) \in \operatorname{dom}(A)$ for all $t \in [0,\infty)$, and u satisfies (ACP).

(c) The abstract Cauchy problem (ACP) is called **classically well-posed** if for every $x \in \operatorname{dom}(A)$ it has a unique classical solution $u(\cdot, x)\colon [0,\infty) \to X$ and for every $\tau \geq 0$ one has $\sup_{s\in[0,\tau]} \|u(s,x)\|_X \to 0$ as $\|x\|_X \to 0$.[2]

(d) Let $x \in X$. A function $u\colon [0,\infty) \to X$ is said to be a **mild solution** of (ACP) if $u \in \mathrm{C}([0,\infty); X)$, one has $\int_0^t u(s)\,\mathrm{d}s \in \operatorname{dom}(A)$ for all $t \in [0,\infty)$, and

$$u(t) = x + A\int_0^t u(s)\,\mathrm{d}s.$$

(e) The abstract Cauchy problem (ACP) is called **mildly well-posed** if for every $x \in X$ it has a unique mild solution $u(\cdot, x)\colon [0,\infty) \to X$, and for every $\tau \geq 0$ one has $\sup_{s\in[0,\tau]} \|u(s,x)\|_X \to 0$ as $\|x\|_X \to 0$.

The definitions of well-posedness above consist of the three properties that one commonly requires of solutions: existence, uniqueness, and continuous dependence on the initial data. It turns out that continuous dependence is actually redundant in the definition of mild well-posedness, as Theorem 10.1.7 below shows.

Before characterising well-posedness of abstract Cauchy problems, we discuss properties of solutions in the following propositions.

Proposition 10.1.2. *Let $A\colon X \supseteq \operatorname{dom}(A) \to X$ be a closed operator on a Banach space X and let $x \in X$. If $u\colon [0,\infty) \to X$ is a classical solution to* (ACP)*, then $x \in \operatorname{dom}(A)$ and u is a mild solution.*

Proof. Since u is a classical solution, $x = u(0) \in \operatorname{dom}(A)$. We can then integrate the differential equation in (ACP) to obtain

$$u(t) = x + \int_0^t Au(s)\,\mathrm{d}s = x + A\int_0^t u(s)\,\mathrm{d}s;$$

the second equality follows from Theorem 4.A.10, since A is closed by assumption. □

[1] This definition is not always consistent in the literature, e.g. a commonly used alternative is [Lun13, Definition 4.1.1(iii)].

[2] There is a slight abuse of terminology here: (ACP) is an initial value problem, i.e. it depends on $x \in X$. Hence, classical well-posedness is actually not a property of (ACP), but rather of the family of all abstract Cauchy problems (ACP) for which $x \in \operatorname{dom}(A)$. A similar comment applies to definition of mild well-posedness in part (e) of the definition.

Proposition 10.1.3. *Let $A\colon X \supseteq \operatorname{dom}(A) \to X$ be an operator on a Banach space X. Let $x \in X$ and let λ be a scalar.*

A function $u\colon [0,\infty) \to X$ is a classical (mild) solution to (ACP) if and only if the rescaled function $v := e^{-\lambda \cdot} u$ is a classical (mild) solution to the shifted problem

$$\begin{cases} \dot v(t) = (A-\lambda)v(t), & t \in [0,\infty), \\ v(0) = x. \end{cases}$$

Proof. For classical solutions, the proposition follows by a straightforward differentiation. It is an instructive exercise to prove the corresponding assertion for mild solutions, hence we invite the reader to provide the details in Exercise 10.1. □

Proposition 10.1.4. *Let $A\colon X \supseteq \operatorname{dom}(A) \to X$ be an operator on a Banach space X such that (ACP) is mildly well-posed. For each $x \in X$, let $u(\cdot,x)\colon [0,\infty) \to X$ be the mild solution of (ACP) with initial condition $u(0,x) = x$ and for every $t \in [0,\infty)$, consider the map $T(t)\colon X \to X$, $x \mapsto u(t,x)$. The following properties hold:*

(a) *For each $\tau \in (0,\infty)$, the mapping $X \to \mathrm{C}([0,\tau];X)$, $x \mapsto u(\cdot,x)|_{[0,\tau]}$ is linear and continuous. In particular, $T(t) \in \mathcal{L}(X)$ for each $t \in [0,\infty)$.*

(b) *$T(0) = \mathrm{id}_X$ and $T(t+s) = T(t)T(s) = T(s)T(t)$ for all $t, s \geq 0$.*

(c) *There exist numbers $M > 0$ and $\omega \in \mathbb{R}$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$.*

Proof. (a) Linearity is a consequence of the uniqueness of mild solutions. Indeed, for all $x, y \in X$ and scalars α, β, the function $\alpha u(\cdot,x) + \beta u(\cdot,y)$ is a mild solution of (ACP) with initial value $\alpha x + \beta y$, and thus $\alpha u(\cdot,x) + \beta u(\cdot,y) = u(\cdot,\alpha x + \beta y)$ by uniqueness. The continuous dependence in the definition of mild well-posedness shows the continuity of the map $x \mapsto u(\cdot,x)|_{[0,\tau]}$ at $x = 0$, and hence at every $x \in X$ due to linearity.

(b) For every $x \in X$, we have $T(0)x = u(0,x) = x$, so $T(0) = \mathrm{id}_X$. Next, let $s \geq 0$ and observe that $T(\cdot)T(s)x = u(\cdot,u(s,x))$ is a mild solution of (ACP) with initial condition $u(s,x)$. On the other hand, the function $v_s := u(\cdot + s, x)$ satisfies $v_s(0) = u(s,x)$, and

$$\begin{aligned} A\int_0^t v_s(r)\,\mathrm{d}r = A\int_0^t u(r+s,x)\,\mathrm{d}r = A\int_s^{t+s} u(r,x)\,\mathrm{d}r \\ = \big(u(t+s,x) - x\big) - \big(u(s,x) - x\big) = u(t+s,x) - u(s,x) \end{aligned}$$

for every $t \geq 0$. This shows that v_s is also a mild solution of (ACP) with initial value $u(s,x)$, and thus $u(t+s,x) = u(t,u(s,x))$ by uniqueness, i.e. $T(t+s)x = T(t)T(s)x$.

(c) Let $M := \sup_{s\in[0,1]} \|T(s)\|$. It follows from the continuity of mild solutions and the uniform boundedness principle that $M < \infty$. For each $t \geq 0$, write $t = s + n$ for $n \in \mathbb{N}_0$ and $s \in [0,1)$. Taking $\omega := \log M$, we deduce from (b) that

$$\|T(t)\| = \|T(s)T(n)\| \leq \|T(s)\|\,\|T(1)\|^n \leq M \cdot M^n \leq Me^{\omega t}.$$ □

The operator family $(T(t))_{t\geq 0}$ in Proposition 10.1.4 is central to the study of abstract Cauchy problems and are studied in more detail in Section 10.2.

Remark 10.1.5. In the situation of the preceding proposition, note that for each $x \in X$, the continuity of the mapping $u(\cdot, x)\colon [0,\infty) \to X$, $t \mapsto u(t,x) = T(t)x$ can be phrased in terms of the operator family $(T(t))_{t\geq 0}$ by saying that all its **orbit maps** are continuous.

Proposition 10.1.4(c) shows that, for a mildly well-posed Cauchy problem, mild solutions are exponentially bounded. This justifies $\omega_0(A) < \infty$ in the following definition.

Definition 10.1.6. Let $A\colon X \supseteq \operatorname{dom}(A) \to X$ be an operator on a Banach space X such that the abstract Cauchy problem (ACP) is mildly well-posed. For every $x \in X$ let the function $u(\cdot, x)\colon [0,\infty) \to X$ denote the unique mild solution of (ACP). Then

$$\omega_0(A) := \inf\left\{\omega \in \mathbb{R} : \exists M \geq 1 \text{ with } \|u(t,x)\| \leq Me^{\omega t}\|x\| \;\; \forall\, t \geq 0,\, x \in X\right\} \in [-\infty, \infty)$$

is called the **growth bound** of A.

For densely defined operators, the two notions of well-posedness of (ACP) coincide:

Theorem 10.1.7. *Let $A\colon X \supseteq \operatorname{dom}(A) \to X$ be a closed operator on a complex Banach space X. The following assertions are equivalent.*

(i) (ACP) *is classically well-posed and A is densely defined.*

(ii) (ACP) *is mildly well-posed.*

(iii) *For every $x \in X$,* (ACP) *has a unique mild solution.*

Moreover, if these equivalent conditions are satisfied and $x \in X$, then a mild solution to (ACP) *is a classical solution if and only if the initial value $x \in \operatorname{dom}(A)$.*

Proof. "(i) ⇒ (iii)": *Uniqueness*: Due to the linear dependence on the initial value, it suffices to show that the constant function with value 0 is the only mild solution for the initial value 0. So let u be a mild solution with initial value 0. It can be readily checked by definition that $v\colon [0,\infty) \to X$, $t \mapsto v(t) := \int_0^t u(s)\,\mathrm{d}s$ is a classical solution with initial value 0. Hence, the classical well-posedness implies that $v = 0$. In turn, $u = 0$.

Existence: For each $x \in \operatorname{dom}(A)$, let $u(\cdot, x)\colon [0,\infty) \to X$ denote the unique classical solution to the abstract Cauchy problem (ACP). By uniqueness of classical solutions,

$$S\colon \operatorname{dom}(A) \to \mathrm{C}([0,\infty); X), \qquad x \mapsto u(\cdot, x).$$

is linear. Also, for every $\tau > 0$, the continuity condition in the classical well-posedness implies that the operator $(S\,\cdot\,)|_{[0,\tau]}\colon \left(\operatorname{dom}(A), \|\cdot\|_X\right) \to \mathrm{C}([0,\tau]; X)$ is continuous. Density of $\operatorname{dom}(A)$ yields a linear map $\tilde{S}\colon X \to \mathrm{C}([0,\infty); X)$ such that $(\tilde{S}\,\cdot\,)|_{[0,\tau]}\colon X \to \mathrm{C}([0,\tau]; X)$ is the unique continuous extension of $(S\,\cdot\,)|_{[0,\tau]}$ for each $\tau > 0$.

Now let $x \in X$ and choose a sequence (x_n) in $\operatorname{dom}(A)$ with $x_n \to x$. For each n, it follows from Proposition 10.1.2 that the classical solution Sx_n is also a mild solution. Moreover, $Sx_n \to \tilde{S}x$ uniformly on compact time intervals, so one can derive from the closedness of A that $\tilde{S}x$ is a mild solution.

"(iii) ⇒ (ii)": For each $x \in X$, let $u(\cdot,x)\colon [0,\infty) \to X$ denote the mild solution to the Cauchy problem (ACP) and fix $\tau > 0$. We need to show that

$$S_\tau\colon X \to \mathrm{C}([0,\tau],X), \qquad x \mapsto u(\cdot,x)|_{[0,\tau]}.$$

is continuous. By the closed graph theorem, it suffices to prove that S_τ is a closed operator. Let $(x_n) \subset X$ be a sequence such that $x_n \to x$ in X and $u_n := S_\tau x_n \to v$ in $\mathrm{C}([0,\tau];X)$ as $n \to \infty$. The latter implies that $\int_0^t u_n(s)\,\mathrm{d}s \to \int_0^t v(s)\,\mathrm{d}s$ for every $t \in [0,\tau]$. Since each $S_\tau x_n$ is a mild solution of (ACP) restricted to $[0,\tau]$, we have

$$A\int_0^t u_n(s)\,\mathrm{d}s = u_n(t) - x_n \to v(t) - x$$

for each $t \in [0,\tau]$. The closedness of A implies $\int_0^t v(s)\,\mathrm{d}s \in \operatorname{dom}(A)$ and

$$v(t) - x = A\int_0^t v(s)\,\mathrm{d}s, \qquad t \in [0,\tau].$$

Hence $v = S_\tau x$, and S_τ is indeed closed.

"(ii) ⇒ (i)": For each $x \in X$, let $u(\cdot,x)\colon [0,\infty) \to X$ be the mild solution of (ACP).

A is densely defined: For $x \in X$, the vectors $v_t := \frac{1}{t}\int_0^t u(s,x)\,\mathrm{d}s$ belong to $\operatorname{dom}(A)$ for all $t > 0$, and $v_t \to u(0,x) = x$ as $t \downarrow 0$.

To complete the proof of (i) and of the additional property stated at the end of the theorem, it suffices to show that the mild solution $u(\cdot,x)$ is a classical solution of (ACP) whenever $x \in \operatorname{dom}(A)$. We do this in several steps.

Step 1: Observe that the mild solution $u(\cdot,x)$ is a classical solution under the stronger assumption that $x \in \operatorname{dom}(A)$ and $u(\cdot,x) \in \mathrm{C}([0,\infty),\operatorname{dom}(A))$, where $\operatorname{dom}(A)$ is endowed with a graph norm.

Indeed, under those assumptions, the definition of a mild solution and the closedness of A imply that $u(t,x) = x + \int_0^t Au(s,x)\,\mathrm{d}s$ for all $t \geq 0$. Hence, $u(\cdot,x) \in \mathrm{C}^1([0,\infty),X)$, $\dot{u}(t,x) = Au(t,x)$ for all $t \geq 0$, and $u(0,x) = x$.

Step 2: We use the exponential bound $\|u(t,x)\| \leq Me^{\omega t}\|x\|$ for all $x \in X$ (Proposition 10.1.4(c)) to show that $\lambda - A\colon \operatorname{dom}(A) \to X$ is bijective for some $\lambda \in \mathbb{C}$.

To this end, fix $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$, and consider the map $R_\lambda\colon X \to X$ given by

$$R_\lambda x := \int_0^\infty e^{-\lambda t}u(t,x)\,\mathrm{d}t \qquad \forall\, x \in X,$$

where the integral is a well-defined Bochner integral (recall Theorem 4.A.8), since the integrand is continuous and decays exponentially. The map R_λ is linear and bounded with $\|R_\lambda\| \leq M(\operatorname{Re}\lambda - \omega)^{-1}$.

Set $v := e^{-\lambda\cdot}u$. For each $x \in X$, $\int_0^t v(s,x)\,\mathrm{d}s \in \operatorname{dom}(A)$ and $v(t,x) = x + (A-\lambda)\int_0^t v(s,x)\,\mathrm{d}t$ for all $t > 0$ by Proposition 10.1.3. Moreover, $v(t,x) \to 0$ as $t \to \infty$. It follows from the closedness of $\lambda - A$ that $R_\lambda x \in \operatorname{dom}(A)$ and $(\lambda - A)R_\lambda x = x$.

In particular, $\lambda - A : \operatorname{dom}(A) \to X$ is surjective. For injectivity, let $x \in \ker(\lambda - A)$. Since the constant map $t \mapsto x$ is a mild solution to the shifted problem $\dot{v}(t) = (A-\lambda)v(t), v(0) = x$, Proposition 10.1.3 ensures that $t \mapsto e^{\lambda t}x$ is a mild solution to (ACP). By uniqueness, $u(\cdot, x) = e^{\lambda \cdot} x$. Since $e^{-\lambda t}u(t,x) \to 0$ as $t \to \infty$, so $x = 0$. Thus $\lambda - A$ is indeed bijective, and the proof shows that $\mathcal{R}(\lambda, A) = R_\lambda$.

Step 3: Let $x \in \operatorname{dom}(A)$. We finally show that $u(\cdot, x)$ is a classical solution. By Proposition 10.1.3, it is equivalent to prove this for the shifted problem and $v = e^{-\lambda \cdot} u$. Choose $\lambda \in \mathbb{C}$ as in Step 2. Then $\mathcal{R}(\lambda, A) \in \mathcal{L}(X, \operatorname{dom}(A))$ by the closed graph theorem. Using this, it is straightforward to check that $\mathcal{R}(\lambda, A)v\big(\cdot, (\lambda - A)x\big)$ is also a mild solution to the shifted problem with the initial value x. Moreover, it is in $\mathrm{C}([0,\infty); \operatorname{dom}(A))$, so according to Step 1 it is even a classical solution. Finally, the uniqueness of mild solutions gives $v(\cdot, x) = \mathcal{R}(\lambda, A)v\big(\cdot, (\lambda - A)x\big)$. □

10.2 C_0-semigroups

In Proposition 10.1.4, under the assumption of mild well-posedness, we considered a family of operators $(T(t))_{t\geq 0}$ in $\mathcal{L}(X)$ with the fundamental property that the map $t \mapsto T(t)x$ yields the unique mild solution of (ACP) with initial value x. We now switch perspectives: to answer the question of well-posedness of (ACP), we take the family of operators $(T(t))_{t\geq 0}$ as a starting point.

Definition 10.2.1. Let $(T(t))_{t\geq 0}$ be a family of bounded operators on a Banach space X.

(a) $(T(t))_{t\geq 0}$ is called an **operator semigroup**, or simply a **semigroup**, on X if

$$T(0) = \mathrm{id} \quad \text{and} \quad T(t+s) = T(t)T(s) = T(s)T(t) \quad \forall\, t,s \geq 0.$$

The semigroup $(T(t))_{t\geq 0}$ is called **strongly continuous**, or in short a C_0**-semigroup**, if $T(t)x \to x$ as $t \downarrow 0$ for all $x \in X$.

(b) If $(T(t))_{t\geq 0}$ is a C_0-semigroup on X, its **generator** is the operator $A\colon X \supseteq \operatorname{dom}(A) \to X$ defined by

$$\operatorname{dom}(A) := \left\{ x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists in } X \right\},$$
$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t}.$$

The following result shows that it often suffices to check strong continuity of a semigroup on a 'nice' subset of elements. On the other hand, strong continuity automatically extends from 0 to all other times.

Proposition 10.2.2. *Let $(T(t))_{t\geq 0}$ be an operator semigroup on a Banach space X. The following assertions are equivalent:*

(i) *$(T(t))_{t\geq 0}$ is a C_0-semigroup.*

(ii) *For every $x \in X$, the orbit map $[0,\infty) \to X$, $t \mapsto T(t)x$ is continuous.*

(iii) *There exists $\tau > 0$ such that $\sup_{t\in[0,\tau]} \|T(t)\|_{\mathcal{L}(X)} < \infty$, and there exists a dense subset $D \subseteq X$ such that $T(t)x \to x$ as $t \downarrow 0$ for all $x \in D$.*

Proof. "(i) ⇒ (ii)": The semigroup property and the continuity at $t = 0$ in the definition of a C_0-semigroup readily give that all orbits are continuous from the right at every time $t \in [0,\infty)$. Showing the left continuity at every $t \in (0,\infty)$, is a bit more involved. We do not discuss it here and refer e.g. to [EN00, Proposition I.5.3] instead.

"(ii) ⇒ (iii)": The estimate $\sup_{t\in[0,\tau]} \|T(t)\|_{\mathcal{L}(X)} < \infty$ for all $\tau > 0$ follows from (ii) by the uniform boundedness principle. Moreover, we can simply choose $D := X$.

"(iii) ⇒ (i)": This implication follows by a standard 3ε-argument. □

It turns out that well-posedness of the abstract Cauchy problem (ACP) is equivalent to the operator A being the generator of a C_0-semigroup, and this is in turn equivalent to a resolvent estimate for A.

Theorem 10.2.3 (Generation theorem)**.** *Let $A\colon X \supseteq \operatorname{dom}(A) \to X$ be an operator on a complex Banach space X. The following assertions are equivalent.*

(i) *The operator A is closed and satisfies the equivalent conditions of the well-posedness Theorem 10.1.7.*

(ii) *The operator A is a generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X.*

(iii) *The operator A is closed, densely defined, and there exist $\omega \in \mathbb{R}$ and $M \geq 1$ such that $(\omega,\infty) \subseteq \rho(A)$ and for each $\lambda \in (\omega,\infty)$, we have*

$$\left\|(\lambda-\omega)^n \mathcal{R}(\lambda, A)^n\right\| \leq M \qquad \forall\ n \in \mathbb{N}_0.$$

(iv) *The operator A is closed, densely defined, and there exist $\omega \in \mathbb{R}$ and $M \geq 1$ such that every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ satisfies $\lambda \in \rho(A)$ and*

$$\left\|(\operatorname{Re}\lambda-\omega)^n \mathcal{R}(\lambda, A)^n\right\| \leq M \qquad \forall\ n \in \mathbb{N}_0.$$

Assume now that the equivalent conditions (i)–(iv) are satisfied. Then the C_0-semigroup $(T(t))_{t\geq 0}$ in (ii) is uniquely determined and the following properties hold:

(a) *For every $x \in X$, the mild solution u to (ACP) is given by $u(t) = T(t)x$ for all $t \in [0,\infty)$.*

(b) Laplace transform representation: *If $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0(A)$, then $\lambda \in \rho(A)$ and*

$$\mathcal{R}(\lambda, A)x = \int_0^\infty e^{-\lambda s} T(s)x \,\mathrm{d}s \qquad \forall\ x \in X.$$

Proof. "(i) ⇒ (iv)": By Theorem 10.1.7, $\operatorname{dom}(A)$ is dense. Also, by the proof of (ii) ⇒ (i) of that theorem, there exists $\omega \in \mathbb{R}$ such that each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ satisfies

$$\lambda \in \rho(A) \quad \text{and} \quad \mathcal{R}(\lambda, A)x = \int_0^\infty e^{-\lambda s} T(s)x \, \mathrm{d}s \quad \forall\, x \in X.$$

From the Taylor series representation of the resolvent (Proposition 3.3.2(a)), we have

$$\mathcal{R}(\lambda, A)^n x = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} \mathcal{R}(\lambda, A)x = \int_0^\infty \lambda^{n-1} e^{-\lambda s} T(s)x \, \mathrm{d}s,$$

where the second equality is obtained from the above integral representation plus an induction argument. By the choice of ω, we have $\|T(s)\| \le Me^{\omega s}$ for all $s \ge 0$. In turn,

$$\left\| (\operatorname{Re}\lambda - \omega)^n \mathcal{R}(\lambda, A)^n \right\| \le M \qquad \forall\, n \in \mathbb{N}.$$

"(iv) ⇒ (iii)": This implication is trivial.

"(iii) ⇒ (ii)": This is the most non-trivial part of the proof, since we have to construct the semigroup from the resolvent. The key idea is to define a sequence of *bounded* operators $A_n \in \mathcal{L}(X)$ by

$$A_n := nA\mathcal{R}(n, A) = n^2 \mathcal{R}(n, A) - n. \tag{10.2.1}$$

One can show that $A_n x \to Ax$ for every $x \in \operatorname{dom}(A)$ (you have already seen essentially the same argument in Exercise 5.1). Since A_n is bounded, the exponential function $t \mapsto e^{tA_n}$ can be defined as a series just as for matrices. The tricky part is to prove convergence of $(e^{tA_n})_{t\ge 0}$ to a C_0-semigroup in an appropriate sense. This procedure is called the **Yosida approximation** technique. The details are provided in the supplementary Section 10.A for the interested reader.

"(ii) ⇒ (i)": We show that (ACP) has a unique mild solution for each $x \in X$. So fix $x \in X$.

Existence: Set $u(t, x) := T(t)x$ for all $t \in [0, \infty)$. Then $u(\cdot, x) \in \mathrm{C}([0, \infty); X)$ according to Proposition 10.2.2. Let us show that $u(\cdot, x)$ is a mild solution of (ACP). For $\tau \ge 0$, let $y = \int_0^\tau T(s)x \, \mathrm{d}s$. Then

$$\begin{aligned}\frac{T(t)y - y}{t} &= \frac{1}{t}\int_0^\tau T(t+s)x \, \mathrm{d}s - \frac{1}{t}\int_0^\tau T(s)x \, \mathrm{d}s = \frac{1}{t}\int_t^{t+\tau} T(s)x \, \mathrm{d}s - \frac{1}{t}\int_0^\tau T(s)x \, \mathrm{d}s \\ &= \frac{1}{t}\int_\tau^{t+\tau} T(s)x \, \mathrm{d}s - \frac{1}{t}\int_0^t T(s)x \, \mathrm{d}s\end{aligned}$$

converges to $T(\tau)x - x$ as $t \downarrow 0$. In particular, $y \in \operatorname{dom}(A)$ and $Ay = T(\tau)x - x$. This shows that $u(t, x) = T(t)x$ is indeed a mild solution of (ACP).

Uniqueness: Let $w \in C([0, \infty); X)$ be another mild solution of (ACP). By linearity, it suffices to assume that $x = 0$. Fix $t > 0$. For $v := \int_0^{t-\cdot} T(s)w(\cdot) \, \mathrm{d}s$, it can be shown that $v' = -w$ and $v(t) = v(0) = 0$. Now $w \equiv 0$ because

$$\int_0^t w(s) \, \mathrm{d}s = -\int_0^t v'(s) \, \mathrm{d}s = -v(t) = 0.$$

A is closed: Let $(x_n) \subseteq \operatorname{dom}(A)$ and $x, y \in X$ be such that $x_n \to x$ and $Ax_n \to y$. Fix $n \in \mathbb{N}$ and $\tau > 0$. Then

$$\lim_{t \downarrow 0} T(s) \frac{T(t)x_n - x_n}{t} = T(s)Ax_n$$

uniformly for $s \in [0, \tau]$ and hence

$$\lim_{t \downarrow 0} \frac{T(t) - \mathrm{id}}{t} \int_0^\tau T(s)x_n \, \mathrm{d}s = \lim_{t \downarrow 0} \int_0^\tau T(s) \frac{T(t)x_n - x_n}{t} \, \mathrm{d}s = \int_0^\tau T(s)Ax_n \, \mathrm{d}s.$$

As shown above, $u(\cdot, x) = T(\cdot)x$ is a mild solution of (ACP). Hence,

$$T(t)x_n - x_n = A \int_0^t T(s)x_n \, \mathrm{d}s = \int_0^t T(s)Ax_n \, \mathrm{d}s$$

and in turn $T(t)x - x = \int_0^t T(s)y \, \mathrm{d}s$ for all $t > 0$. It follows that $x \in \operatorname{dom}(A)$ and $Ax = T(0)y = y$. Thus, A is closed.

Consequences of (i)–(iv)*:* Assertions (a) and (b) were shown in the proofs of (ii) ⇒ (i) and (i) ⇒ (iv), respectively. The uniqueness of the semigroup follows from the uniqueness of classical solutions in the well-posedness Theorem 10.1.7. □

Examples 10.2.4. We now look at some examples of semigroups and their generators.

(a) *Matrix semigroups:* By now, the reader is well-acquainted with the matrix semigroup $T(t) := e^{tA}$, where $A \in \mathbb{C}^{n \times n}$ is a given matrix. Evidently, the semigroup $(e^{tA})_{t \geq 0}$ on $\mathbb{C}^n$ is strongly continuous; in fact, from the exponential series one even obtains the stronger property $\lim_{t \downarrow 0} \|e^{tA} - I\| = 0$, which is often called **uniform continuity** in operator semigroup theory. Unsurprisingly, since $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} e^{tA} = A$, the generator of $(e^{tA})_{t \geq 0}$ is A.

(b) *Semigroups with bounded generator:* Let X be a Banach space and let $A \in \mathcal{L}(X)$. Note that e^A makes sense if defined by the exponential series. If we set $T(t) := e^{tA}$ for all $t \geq 0$, then $(T(t))_{t \geq 0}$ is a C_0-semigroup on X with generator A. This situation closely resembles the finite-dimensional case and the properties we just claimed can be shown by similar arguments as in Section 1.3. It turns out that $[0, \infty) \to \mathcal{L}(X)$, $t \mapsto e^{tA}$ is even continuous with respect to the operator norm.[3]

(c) *Left shift semigroup:* Consider the Banach space $X := \mathrm{BUC}(\mathbb{R})$ consisting of bounded, uniformly continuous functions $f : \mathbb{R} \to \mathbb{C}$, equipped with the supremum norm. The **left shift semigroup** $(T(t))_{t \geq 0}$ on X is defined by

$$T(t)f = f(\cdot + t) \qquad \forall f \in X, \; t \geq 0.$$

It is a C_0-semigroup on X and its A generator is given by

$$\operatorname{dom}(A) := \{f \in X : f \text{ differentiable with } f' \in X\}, \qquad Af := f'.$$

[3] In fact, every C_0-semigroup that is continuous with respect to the operator norm is of this form; see e.g. [EN00, Theorem I.3.7].

Proof. The semigroup property is obvious. To show the strong continuity, let $f \in X$. Since f is uniformly continuous on $\mathbb{R}$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f(\cdot + s) - f\|_\infty < \varepsilon$ for all $s \in \mathbb{R}$ with $|s| < \delta$. This implies that $T(t)f \to f$ as $t \downarrow 0$, so $(T(t))_{t\geq 0}$ is indeed a C_0-semigroup.

Let $B\colon \mathrm{BUC}(\mathbb{R}) \supseteq \mathrm{dom}(B) \to \mathrm{BUC}(\mathbb{R})$ denote the generator of $(T(t))_{t\geq 0}$. We first show that $\mathrm{dom}(B) \subseteq \mathrm{dom}(A)$ and that $Bf = Af$ for each $f \in \mathrm{dom}(B)$. So let $f \in \mathrm{dom}(B)$. Then in particular, for every $x \in \mathbb{R}$, the limit $\lim_{t\downarrow 0} t^{-1}(f(x+t)-f(x))$ exists and is equal to $(Bf)(x) \in X$. Thus f is differentiable and $f' = Bf \in X$. Consequently $f \in \mathrm{dom}(A)$ and $Af = Bf$.

To conclude that $A = B$ it suffices to show that $\rho(A) \cap \rho(B) \neq \emptyset$.[4] As $\|T(t)\|_{\mathcal{L}(X)} \leq 1$ for all $t \geq 0$, the Laplace transform representation of the resolvent (see Theorem 10.2.3(b)) implies that $1 \in \rho(B)$. On the other hand, given $g \in X$, we can simply solve the ODE $u - u' = g$ under the condition that u is bounded on $\mathbb{R}$ and obtain

$$u(x) = \int_x^\infty e^{-(y-x)} g(y)\,\mathrm{d}y = \int_0^\infty e^{-z} g(z+x)\,\mathrm{d}z \qquad \forall x \in \mathbb{R}.$$

The second formula above easily yields the uniform continuity of u, and then $u' = u - g$ is also bounded and uniformly continuous. This shows that $u \in \mathrm{dom}(A)$, $1 \in \rho(A)$ and $u = \mathcal{R}(1, A)g$. Therefore $1 \in \rho(A) \cap \rho(B)$. □

Motivated by Example 10.2.4(b) and also the fact that a C_0-semigroup $(T(t))_{t\geq 0}$ is uniquely determined by its generator A (Theorem 10.2.3), one often uses the notation $T(t) =: e^{tA}$ for $t \geq 0$. [5]

In the previous examples, it was easy to describe the semigroup explicitly, and hence determine the generator. However, one often has the converse situation: a concrete operator A is given, and we want to know if it is a generator. It can be quite impractical to check conditions (iii) or (iv) in Theorem 10.2.3 directly. A notable exception is when the operator A comes from a sesquilinear form.

Theorem 10.2.5. *Let the Hilbert spaces V, H and the sesquilinear form $\mathfrak{a}\colon V \times V \to \mathbb{C}$ satisfy the assumptions of Theorem 5.1.4. Then the associated operator $A\colon H \supseteq \mathrm{dom}(A) \to H$ generates a C_0-semigroup on H.*

Proof. By Theorem 5.1.4, A is closed and densely defined, and there exists $\mu \in \mathbb{R}$ such that every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > \mu$ satisfies $\lambda \in \rho(A)$ and $\|(\mathrm{Re}\,\lambda - \mu)\mathcal{R}(\lambda, A)\|_{H\leftarrow H} \leq 1$. So, condition (iv) in Theorem 10.2.3 holds, and hence A generates a C_0-semigroup on H. □

Example 10.2.6. Let us revisit the Laplacian with non-local Robin boundary conditions which was studied in Exercise 5.6. Recall that $\Delta_B\colon \mathrm{dom}(\Delta_B) \subseteq L^2(0,1) \to L^2(0,1)$ is the

[4] More generally, it is easily verified that if an injective operator T extends a surjective operator S, then in fact $S = T$.

[5] Note, however, that this is indeed just a notation. By no means does it imply that e^{tA} could be represented as a series using powers of A.

operator associated to the sesquilinear form $\mathfrak{a}\colon H^1(0,1)\times H^1(0,1)\to\mathbb{C}$ defined by

$$\mathfrak{a}(v,u) = \int_0^1 \overline{v'}u'\,\mathrm{d}x - \begin{pmatrix}\overline{v(0)} & \overline{v(1)}\end{pmatrix} B \begin{pmatrix}u(0)\\ u(1)\end{pmatrix};$$

for a given matrix $B\in\mathbb{R}^{2\times 2}$. It is easy to see that $\mathfrak{a}$ is a bounded sesquilinear form on $H^1(0,1)$. Moreover, by Exercise 5.6(b) there exist numbers $\mu\in\mathbb{R}$ and $c>0$ such that

$$\operatorname{Re}\mathfrak{a}(u,u) + \mu\|u\|^2_{L^2(0,1)} \ge c\|u\|^2_{H^1(0,1)} \qquad \forall\, u\in H^1(0,1).$$

All the conditions of Theorem 5.1.4 are therefore satisfied, so by Theorem 10.2.5 we deduce that Δ_B generates a C_0-semigroup on $L^2(0,1)$.

10.3 Positive semigroups on Banach lattices

At the beginning of the course, we already studied positive matrix semigroups (Definition 1.3.4). We now come full circle, and discuss the analogous concept for infinite-dimensional linear dynamical systems.

Definition 10.3.1. An operator semigroup $(T(t))_{t\ge 0}$ on a Banach lattice E is called **positive** if e^{tA} is a positive operator on E for each $t\ge 0$.

Examples 10.3.2. Let us discuss which of the semigroups in Examples 10.2.4 are positive.

(a) In finite dimensions, we understand the situation well: various characterisations of positivity of matrix semigroups were given in Theorem 1.3.8 and Exercise 1.2.

(b) Let E be a Banach lattice and $A\in\mathcal{L}(E)$. The C_0-semigroup $(e^{tA})_{t\ge 0}$ is positive if and only if there exists $\omega\in\mathbb{R}$ such that $A+\omega\ge 0$.

The implication "⇐" is easy to see by the same argument as in the finite-dimensional case: one has $e^{tA} = e^{-t\omega}e^{t(A+\omega)}\ge 0$ for all $t\ge 0$. The converse implication "⇒" is a bit more involved on Banach lattices and we do not prove it here – see e.g. [AGG⁺86, Theorem C.II-1.11] for a proof.

(c) The Banach space BUC($\mathbb{R}$) is a Banach lattice, with respect to the pointwise order on its real part. The left shift semigroup on BUC($\mathbb{R}$) is evidently positive.

Just as in the finite dimensional case (Theorem 1.3.8), positivity of semigroups can be characterised via the resolvent.

Proposition 10.3.3. *A C_0-semigroup $(e^{tA})_{t\ge 0}$ on a complex Banach lattice E is positive if and only if $\mathcal{R}(\lambda,A)\ge 0$ for sufficiently large $\lambda\in\rho(A)\cap\mathbb{R}$.*

Proof. "⇒": If $(e^{tA})_{t\ge 0}$ is positive, then the Laplace transform representation of the resolvent from Theorem 10.2.3(b) immediately yields the positivity of the resolvent $\mathcal{R}(\lambda,A)$ for all sufficiently large $\lambda\in\mathbb{R}$.

"$\Leftarrow$": This implication can be shown using the approximating operators (10.2.1). Indeed, if $\mathcal{R}(n,A) \geq 0$ for all sufficiently large $n \in \mathbb{N}$, then

$$e^{tA_n} = e^{tn^2\mathcal{R}(n,A)}e^{-nt} \geq 0 \qquad \forall\ t \geq 0.$$

It can then be shown that $e^{tA_n}x \to e^{tA}x$ as $n \to \infty$ for all $x \in X$ uniformly on compact time intervals, and thus $e^{tA} \geq 0$ for all $t \geq 0$; see Section 10.A for details. □

Proposition 10.3.3 combined with the Beurling-Deny criterion for resolvents leads to a very useful characterisation of positive semigroups arising from sesquilinear forms.

Corollary 10.3.4 (The Beurling–Deny criterion for semigroups)**.** *Under the assumptions of Theorem 5.1.4, let $H = L^2(\Omega,\nu)$ for a σ-finite measure space (Ω,ν). If $\mathfrak{a}$ is real, then the following are equivalent for the associated operator $A\colon L^2(\Omega,\nu) \supseteq \operatorname{dom}(A) \to L^2(\Omega,\nu)$:*

(i) *A generates a positive C_0-semigroup on H.*

(ii) *$V_{\mathbb{R}} := V \cap L^2(\Omega,\nu;\mathbb{R})$ is a sublattice of $L^2(\Omega,\nu;\mathbb{R})$ and $\mathfrak{a}(v^-,v^+) \leq 0$ for all $v \in V_{\mathbb{R}}$.*

Proof. By Theorem 10.2.5, we know that A generates a C_0-semigroup on H. Therefore, the equivalence is a consequence of Proposition 10.3.3 and the Beurling–Deny criterion for positivity of resolvents in Theorem 5.1.7. □

Example 10.3.5. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open. The Dirichlet Laplacian

$$\Delta_{\text{Dir}}\colon L^2(\Omega) \supseteq \operatorname{dom}(\Delta_{\text{Dir}}) \to L^2(\Omega)$$

generates a positive C_0-semigroup.

Proof. We know that Δ_{Dir} is associated to the symmetric bounded form $\mathfrak{a}(v,u) = (\nabla v \mid \nabla u)_{L^2}$ on $\operatorname{dom}(\mathfrak{a}) = H_0^1(\Omega)$; see Example 3.3.6(b). For each $\mu > 0$, we have

$$\operatorname{Re}\mathfrak{a}(v,v) + \mu\,\|v\|_{L^2}^2 = \|\nabla v\|_{L^2}^2 + \mu\,\|v\|_{L^2}^2 \geq \min\{1,\mu\}\,\|v\|_{H_0^1}^2.$$

Moreover, we know from Exercise 4.3 that $\mathcal{R}(\lambda,A) \geq 0$ for all $\lambda > 0$. Therefore by Theorem 10.2.5 and Proposition 10.3.3, Δ_{Dir} generates a positive C_0-semigroup. □

Example 10.3.6. Consider Δ_B from Example 10.2.6 again. In Exercise 5.6(c), we showed that $\mathcal{R}(\lambda,\Delta_B) \geq 0$ for all sufficiently large $\lambda \in \mathbb{R}$ if and only if all off-diagonal entries of the matrix B are positive, i.e. $B_{jk} \geq 0$ for all indices $j \neq k$. Hence Corollary 10.3.4 shows that the same condition on B characterises the positivity of the semigroup generated by Δ_B.

In Example 5.4.3, we considered the specific choice

$$B = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

whose off-diagonal entries are evidently not positive. Hence the corresponding semigroup is not positive. However, we will come back to this semigroup later to see that it still has some positivity properties.

Exercises for Chapter 10

Exercise 10.1. Complete the proof of Proposition 10.1.3: Let X be a Banach space, and fix $x \in X$ and a scalar λ. Show that $u \in \mathrm{C}([0,\infty); X)$ is a mild solution of (ACP) with initial value x if and only if $v := e^{-\lambda \cdot} u$ is a mild solution of the shifted abstract Cauchy problem

$$\begin{cases} \dot{v}(t) = (A - \lambda) v(t), & t \in [0,\infty), \\ v(0) = x. \end{cases}$$

Exercise 10.2. Let X be a Banach space and $B\colon \operatorname{dom}(B) \subseteq X \to X$ an unbounded operator. Consider the operator $\mathcal{A}\colon \operatorname{dom}(\mathcal{A}) \subseteq X \times X \to X \times X$ defined by

$$\operatorname{dom}(\mathcal{A}) := X \times \operatorname{dom}(B), \qquad \mathcal{A} := \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}.$$

(a) Show that $u\colon [0,\infty) \to X \times X$ defined by

$$u(t) := \begin{pmatrix} x + tBy \\ y \end{pmatrix}, \qquad t \geq 0$$

is the unique classical solution of the abstract Cauchy problem associated to $\mathcal{A}$ with initial condition $(x \quad y)^{\mathrm{T}} \in \operatorname{dom}(\mathcal{A})$.

(b) Does there exist a unique mild solution for every $(x \quad y)^{\mathrm{T}} \in X \times X$?

(c) Determine whether $\mathcal{A}$ generates a C_0-semigroup on $X \times X$.

Exercise 10.3. Let $p \in [1,\infty)$. Recall that for two functions $k \in L^1(\mathbb{R})$ and $f \in L^p(\mathbb{R})$, the **convolution** $k * f \in L^p(\mathbb{R})$ is given by

$$(k * f)(x) = \int_{\mathbb{R}} k(x-y) f(y) \, \mathrm{d}y$$

for almost all $x \in \mathbb{R}$. It is a standard fact in functional analysis that the convolution $*$ is associative and satisfies Young's inequality $\|k * f\|_{L^p} \leq \|k\|_{L^1} \|f\|_{L^p}$.

For each $t > 0$, we define

$$k_t\colon \mathbb{R} \to \mathbb{R}, \qquad k_t(x) := \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right).$$

(a) Show that $k_t \in L^1(\mathbb{R})$ and $\|k_t\|_{L^1} = 1$ for each $t > 0$.

(b) Show that $k_s * k_t = k_{s+t}$ for all $s, t > 0$.[6]

(c) The **heat semigroup** $(T(t))_{t\geq 0}$ on $L^p(\mathbb{R})$ is defined by $T(0) = \mathrm{id}$ and

$$T(t)f := k_t * f \qquad \forall\, f \in L^p(\mathbb{R}),\ t > 0.$$

Show that $(T(t))_{t\geq 0}$ is a positive C_0-semigroup.[7]

Hint for the strong continuity: Use Proposition 10.2.2 and choose the set D there as the space of test functions.

Exercise 10.4. Let K be a compact metric space. Suppose $A\colon \operatorname{dom}(A) \subseteq \mathrm{C}(K) \to \mathrm{C}(K)$ is a closed and densely-defined operator such that $[0,\infty) \subseteq \rho(A)$ and $\mathcal{R}(\lambda, A) \geq 0$ for all $\lambda \geq 0$. Let $\mathbb{1}$ denote the constant 1 function on K and set $u := \mathcal{R}(0, A)\,\mathbb{1}$.

(a) Show that the principal ideal $\mathrm{C}(K)_u$ is equal to $\operatorname{dom}(A)$ and conclude that u is a quasi-interior point of $\mathrm{C}(K)_+$. Then derive from Exercise 7.1 that $u \succeq \mathbb{1}$.

(b) Prove that the gauge norm $\|\cdot\|_{\mathrm{C}(K)_u}$ is an equivalent norm on $\mathrm{C}(K)$.

(c) Show $\lambda\mathcal{R}(\lambda, A)u \leq u$ for all $\lambda \geq 0$ and conclude that $\left\|\lambda\mathcal{R}(\lambda, A)f\right\|_{\mathrm{C}(K)_u} \leq \left\|f\right\|_{\mathrm{C}(K)_u}$ for all $\lambda \geq 0$ and all $f \in \mathrm{C}(K)$.

(d) Prove that A is the generator of a positive C_0-semigroup on $\mathrm{C}(K)$.

[6]This is possible by a direct computation. No Fourier transform needed!

[7]The generator of the heat semigroup turns out to be the Laplace operator. We refrain from discussing this in more detail at this point.

Notes for Chapter 10

> I hail a semigroup when I see one, and I seem to see them everywhere! Friends have observed, however, that there are mathematical objects which are not semigroups.[8]
> — Einar Hille

The modern theory of C_0-semigroups originates in the late 1940s with the seminal and independent works of Hille [Hil48] and [Yos48]. The core result, now known as the **Hille-Yosida generation theorem** (Theorem 10.2.3 without the additional connection to Theorem 10.1.7), provides a complete characterisation of generators of strongly continuous semigroups on Banach spaces. Historically, however, this theorem should be viewed not as an isolated development, but as a natural extension of earlier ideas.

Indeed, the conceptual predecessor of semigroup generation theory is Stone's theorem on one-parameter unitary groups on Hilbert spaces [Sto32]. Stone's result identifies self-adjoint operators as precisely the generators of unitary C_0-groups, and it already contains the essential paradigm: spectral information about an operator determines time evolution. Yosida himself explicitly acknowledges this lineage, noting that the "basic result of semi-group theory may be considered a natural generalization of the theorem of M.H. Stone" [Yos95, p. 243].

Several classical monographs from the late 1970s and 1980s helped establish semigroup theory as a standard tool in evolution equations and partial differential equations: Davies [Dav80], Pazy [Paz83], and Goldstein [Gol85]. Each emphasises different aspects of the theory – spectral methods, evolution equations, or applications – but together they shaped what is now regarded as the classical framework. A modern and comprehensive treatment is provided by Engel and Nagel [EN00, EN06]. The book [EN00] also has historical significance as being based on the 1st and the 2nd Internet Seminar. Readers interested in the conceptual origins of semigroup methods may consult the brief historical discussion of the exponential function in [EN00, Chapter VII].

Positive semigroups

The study of positive semigroups has been intertwined with semigroup theory from its earliest days, driven initially by applications to probability theory and diffusion processes. Foundational contributions in this direction include the works of Feller [Fel52]

[8] From the foreword to the treatise [Hil48].

and Dynkin [Dyn56], where positivity arises naturally from Markov processes and elliptic or parabolic equations.

The abstract theory of positive semigroups on Banach lattices is developed in full generality in the monograph [AGG+86], affectionately referred to as the *typewriter book* and the more recent book by Bátkai, Kramar Fijavž, and Rhandi [BKFR17] – a byproduct of the 17th Internet Seminar. The result of Exercise 10.4 corresponds to [AGG+86, Theorem B.II-1.8]. An alternate elegant yet very Banach lattice heavy proof of Example 10.3.2(b) can be found in [Voi88, Proposition 2.1].

Encore: if you want to know more...

10.A Yosida approximation

In this supplementary section, we complete the proof of the implication (iii) $\Rightarrow$ (ii) in Theorem 10.2.3.

Proposition 10.A.1. *Let $A\colon \operatorname{dom}(A) \subseteq X \to X$ be a closed, densely defined operator on a complex Banach space X. Suppose there exist numbers $\omega \in \mathbb{R}$ and $M \geq 1$ such that $(\omega, \infty) \subseteq \rho(A)$ and for each $\lambda \in (\omega, \infty)$, we have*

$$\left\| (\lambda - \omega)^n \mathcal{R}(\lambda, A)^n \right\| \leq M \qquad \forall\, n \in \mathbb{N}.$$

Then A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X.

Proof. By making ω larger if necessary, we may assume that $\omega > 0$.

Observe that for each bounded operator $S \in \mathcal{L}(X)$, the exponential series

$$e^{tS} := \sum_{k=0}^{\infty} \frac{t^k S^k}{k!}$$

converges absolutely in $\mathcal{L}(X)$. It is easy to check that $(e^{tS})_{t\geq 0}$ is a C_0-semigroup with generator S.

Step 1: For $n \in \mathbb{N}$ with $n > \omega$, consider the **Yosida approximants**

$$A_n := nA\mathcal{R}(n, A) = n^2 \mathcal{R}(n, A) - n \in \mathcal{L}(X).$$

Since A is densely defined, $A_n x = n\mathcal{R}(n, A)Ax \to Ax$ as $n \to \infty$ for all $x \in \operatorname{dom}(A)$ by Exercise 5.1(a).

Step 2: We show that e^{tA_n} is uniformly bounded for $n \geq 2\omega$ and for t in any bounded time interval. Indeed, for $t \geq 0$ we have

$$\begin{aligned} \left\| e^{tA_n} \right\| = e^{-nt} \left\| e^{n^2 \mathcal{R}(n,A)} \right\| &= e^{-nt} \left\| \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(n^2 \mathcal{R}(n, A)\right)^k \right\| \\ &\leq M e^{-nt} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{n^{2k}}{(n-\omega)^k} = M e^{-nt} e^{\frac{n^2 t}{n-\omega}} = M e^{2\omega t} \end{aligned}$$

whenever $n \geq 2\omega$.

Step 3: Let $x \in X$ and $\tau > 0$. We claim that $(e^{\cdot A_n} x)_{n \geq 2\omega}$ is a Cauchy sequence in $\mathrm{C}([0,\tau]; X)$. To show this, we first assume that $x \in \operatorname{dom}(A)$. For fixed $t \in [0,\tau]$, let

$$u_{m,n}(s)x = e^{(t-s)A_m} e^{sA_n} x, \qquad s \in [0, t]$$

with $m, n \geq 2\omega$. Then

$$\begin{aligned} \left\| (e^{tA_n} - e^{tA_m})x \right\| &= \int_0^t \left\| u'_{m,n}(s)x \right\| \mathrm{d}s = \int_0^t \left\| u_{m,n}(s)(A_n - A_m)x \right\| \mathrm{d}s \\ &\leq \int_0^t M^2 \left\| (A_n - A_m)x \right\| e^{2\omega t} \mathrm{d}s \leq M^2 \tau \left\| (A_n - A_m)x \right\| e^{2\omega\tau} \to 0 \end{aligned}$$

as $m, n \to \infty$ by Step 1. The claim for arbitrary $x \in X$ now follows from the density of $\operatorname{dom}(A)$ in X and from the estimate shown in Step 2.

Step 4: By Step 3, for each $x \in X$ and each $t \geq 0$ the limit

$$T(t)x := \lim_{n\to\infty} e^{tA_n} x$$

exists, with uniform convergence on bounded time intervals. Thus, $t \mapsto T(t)x$ is continuous at 0 and $T(0)x = x$. In fact, it is also easy to see that $T(t+s)x = T(t)T(s)x = T(s)T(t)x$ for all $t, s \geq 0$. Consequently, $(T(t))_{t\geq 0}$ is a C_0-semigroup.

Step 5: Let $B\colon X \supseteq \operatorname{dom}(B) \to X$ be the generator of $(T(t))_{t\geq 0}$ and $x \in \operatorname{dom}(A)$. From the proof of (ii) ⇒ (i) in Theorem 10.2.3, we know that $e^{\cdot A_n} x$ is a mild solution of the Cauchy problem associated with A_n. As A_n is bounded, we therefore obtain

$$e^{tA_n} x - x = \int_0^t e^{sA_n} A_n x \, \mathrm{d}s.$$

Taking $n \to \infty$ yields $T(t)x - x = \int_0^t T(s)Ax \, \mathrm{d}s$. In turn, $\lim_{t\downarrow 0} t^{-1}(T(t)x - x)$ exists and is equal to Ax. Consequently, $x \in \operatorname{dom}(B)$ with $Bx = Ax$, so B is an extension of A.

Since C_0-semigroups yield well-posed abstract Cauchy problems, it follows from the proof of (i) ⇒ (iv) in Theorem 10.2.3 that $(\omega, \infty) \subseteq \rho(B)$. Thus, for all $\lambda > \omega$, the operators $\lambda - A$ and $\lambda - B$ are bijective. Consequently, $\operatorname{dom}(A) = \operatorname{dom}(B)$ and $B = A$. □

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