Chapter 9

Necessary conditions for eventual positivity and negativity

In the Chapters 7 and 8 you have seen sufficient criteria and characterisations of eventual positivity and negativity of resolvents in terms of spectral conditions. To get eventual positivity (and negativity) of a resolvent at a spectral value λ_0 from the corresponding eigenspaces of A and A', we needed some kind of *domination condition*, namely the condition $\operatorname{dom}(A) \subseteq E_u$ in Theorem 7.3.6 and the condition $\pm\mathcal{R}(\lambda_1, A) \preceq u \otimes \varphi$ in Theorem 8.3.2(3). Even before, we demonstrated in Theorem 6.4.4 that such domination conditions are sometimes necessary to get eventual positivity and negativity jointly.

Theorem 6.4.4 is, however, far from optimal: it assumes uniform eventual positivity and negativity and only gives a conclusion on $\operatorname{dom}(A)$. In this chapter, we will use the techniques developed in the previous two chapters to get much stronger results.

9.1 The uniform case

Our first main result in this chapter concerns necessary conditions for uniform eventual positivity and negativity of resolvents. The proof is very similar to that of Theorem 6.4.4.

Theorem 9.1.1. *Let $A \colon E \supseteq \operatorname{dom}(A) \to E$ be a closed, densely defined, and real operator on a complex Banach lattice E. Assume $\lambda_0 \in \sigma(A) \cap \mathbb{R}$ is an isolated spectral value of A. Let $u \in E_+$ and $\varphi \in E'_+$ and assume:*

(1) $\operatorname{dom}(A^{m_1}) \subseteq E_u$ *for some $m_1 \in \mathbb{N}$.*

(2) $\operatorname{dom}\big((A')^{m_2}\big) \subseteq (E')_\varphi$ *for some $m_2 \in \mathbb{N}$.*

(3) $\mathcal{R}(\cdot, A)$ *is uniformly eventually positive and negative with respect to* 0 *at* λ_0.

Then $\pm\mathcal{R}(\nu, A) \preceq u \otimes \varphi$ for all $\nu \in \rho(A) \cap \mathbb{R}$.

Proof. Set $m := m_1 + m_2$ and choose points $\lambda, \mu \in \rho(A) \cap \mathbb{R}$ such that $\lambda < \lambda_0 < \mu$ and $\mathcal{R}(\lambda, A) \le 0 \le \mathcal{R}(\mu, A)$. By assumptions (1) and (2) one has

$$\operatorname{rg} \mathcal{R}(\mu, A)^{m_1} \subseteq E_u \qquad \text{and} \qquad \operatorname{rg}\big(\mathcal{R}(\mu, A)'\big)^{m_2} \subseteq (E')_\varphi,$$

so it follows from Corollary 8.2.4 that $\pm\mathcal{R}(\lambda, A)\mathcal{R}(\mu, A)^m \preceq u \otimes \varphi$. Now we proceed precisely as in the proof of Theorem 6.4.4. For $S := \mathcal{R}(\lambda, A)(\mu-\lambda)^m\mathcal{R}(\mu, A)^m$ the finite resolvent expansion from Lemma 6.4.5 gives

$$0 \le -\mathcal{R}(\lambda, A) = -\sum_{k=1}^{m}(\mu-\lambda)^{k-1}\mathcal{R}(\mu, A)^k - S \le -S \preceq u \otimes \varphi.$$

So $\pm\mathcal{R}(\lambda, A) \preceq u \otimes \varphi$ and thus, by Proposition 8.3.1, the same estimate is true at every other point $\nu \in \rho(A) \cap \mathbb{R}$. □

Remark 9.1.2. Let all assumptions of Theorem 9.1.1 be satisfied. Then we can apply Theorem 6.4.4 to the ideal $I := E_u$ to obtain $\operatorname{dom}(A) \subseteq E_u$. Moreover, we can also apply Theorem 6.4.4 to the dual operator A' to obtain $\operatorname{dom}\left(A'\right) \subseteq (E')_\varphi$.

Note that the conclusions $\operatorname{dom}(A) \subseteq E_u$ and $\operatorname{dom}\left(A'\right) \subseteq (E')_\varphi$ imply $\pm\mathcal{R}(\nu, A)^2 \preceq u\otimes\varphi$ for all $\nu \in \rho(A) \cap \mathbb{R}$ by Corollary 8.2.4, but they do not directly imply that stronger conclusion $\pm\mathcal{R}(\nu, A) \preceq u \otimes \varphi$ of Theorem 9.1.1. In other words, the latter theorem is not an immediate consequence of Theorem 6.4.4; instead, one needs to repeat the arguments from the proof of Theorem 6.4.4, as we have done above.

An example of an operator that satisfies $\operatorname{dom}(A) \subseteq E_u$ and $\operatorname{dom}\left(A'\right) \subseteq (E')_\varphi$, but not $\pm\mathcal{R}(\nu, A) \preceq u \otimes \varphi$, is the Dirichlet Laplace operator on an interval, where u and φ are chose to be eigenfunctions of the spectral bound $\mathrm{s}(\Delta_{\mathrm{Dir}})$; see the last part of the proof of Example 9.3.1 below.

One can combine the previous result with the sufficient condition in Theorem 8.3.2 to get the following characterisation result.

Corollary 9.1.3. *Let $A\colon E \supseteq \operatorname{dom}(A) \to E$ be a closed, densely defined, and real operator on a complex Banach lattice E. Let $\lambda_0 \in \sigma(A) \cap \mathbb{R}$ be a pole of the resolvent $\mathcal{R}(\cdot, A)$. Let $u \in E_+$ and $\varphi \in E'_+$ and assume that $\operatorname{dom}(A^{m_1}) \subseteq E_u$ and $\operatorname{dom}\left((A')^{m_2}\right) \subseteq (E')_\varphi$ for some $m_1, m_2 \in \mathbb{N}$. Then the following assertions are equivalent:*

(i) *$\mathcal{R}(\cdot, A)$ is uniformly eventually positive and negative with respect to $u \otimes \varphi$ at λ_0.*

(ii) *$\ker(\lambda_0 - A)$ is spanned by a vector $v \succeq u$, the eigenspace $\ker(\lambda_0 - A')$ contains a functional $\psi \succeq \varphi$, and there exists $\lambda_1 \in \rho(A) \cap \mathbb{R}$ such that $\pm\mathcal{R}(\lambda_1, A) \preceq u \otimes \varphi$.*

(iii) *The spectral projection P associated to λ_0 satisfies $P \succeq u \otimes \varphi$ and there exists a number $\lambda_1 \in \rho(A) \cap \mathbb{R}$ such that $\pm\mathcal{R}(\lambda_1, A) \preceq u \otimes \varphi$.*

If the equivalent assertions (i)–(iii) hold, then $\pm\mathcal{R}(\nu, A) \preceq u \otimes \varphi$ for all $\nu \in \rho(A) \cap \mathbb{R}$.

Proof. From the result of Exercise 9.1, the domination assumptions imply that $u \in E_+$ is a quasi-interior point and $\varphi \in E'_+$ is a strictly positive functional.

"(i) ⇒ (iii)": By Theorem 7.3.6, $P \gneq 0$ and λ_0 is a first order pole of $\mathcal{R}(\cdot, A)$. Thus, by assumption there exists $\mu > \lambda_0$ such that $P = (\mu - \lambda_0)\mathcal{R}(\mu, A)P \succeq u \otimes \varphi$. In addition, Theorem 9.1.1 ensures that $\pm\mathcal{R}(\nu, A) \preceq u \otimes \varphi$ for all $\nu \in \rho(A) \cap \mathbb{R}$.

"(iii) ⇒ (ii)": By Theorem 7.3.6, $\ker(\lambda_0 - A)$ is spanned by a vector $v \succeq u$, $\ker(\lambda_0 - A')$ contains a functional ψ, and λ_0 is a first order pole. Thus, $\psi = P'\psi \succeq \varphi$.

"(ii) ⇒ (i)": Note that v is a quasi-interior point of E_+ and ψ is a strictly positive functional; see Exercise 9.1. This implication thus follows from Theorem 8.3.2. □

9.2 The individual case

Now we show that Theorem 6.4.4 in fact stays true even if the resolvent is only assumed to be individually eventually positive and negative, provided that the ideal I from the theorem is a principal ideal.

Theorem 9.2.1. *Let $A\colon E \supseteq \operatorname{dom}(A) \to E$ be a real, closed operator on a complex Banach lattice E, let $\lambda_0 \in \mathbb{R}$ be an isolated spectral value of A, and let $u \in E_+$. Suppose that the following conditions hold.*

(1) $\operatorname{dom}(A^m) \subseteq E_u$ *for some* $m \in \mathbb{N}$.

(2) $\mathcal{R}(\cdot, A)$ *is individually eventually positive and negative with respect to* 0 *at* λ_0.

Then $\operatorname{dom}(A) \subseteq E_u$.

We would like to argue similarly as in the proofs of Theorems 6.4.4 and 9.1.1. Since we only assume individual eventual positivity now, the numbers λ and μ in the proof of Theorem 6.4.4, with $\lambda < \lambda_0 < \mu$ such that $\mathcal{R}(\lambda, A)f \le 0$ and $\mathcal{R}(\mu, A)f \ge 0$, depend on the choice of the vector $0 \lneq f \in E$. To overcome this obstacle, two additional ingredients are necessary. The first one is the following generalisation of the finite resolvent expansion from Lemma 6.4.5 which deals with the problem that we cannot choose μ uniformly.

Lemma 9.2.2. *Let $A\colon X \supseteq \operatorname{dom}(A) \to X$ be a closed operator on a complex Banach space X and let $\lambda, \mu_1, \dots, \mu_n \in \rho(A)$ for some $n \in \mathbb{N}$. Then*

$$\mathcal{R}(\lambda, A) = \sum_{k=1}^{n} \left(\prod_{j=1}^{k-1} (\mu_j - \lambda) \prod_{j=1}^{k} \mathcal{R}(\mu_j, A) \right) + \mathcal{R}(\lambda, A) \prod_{j=1}^{n} (\mu_j - \lambda) \prod_{j=1}^{n} \mathcal{R}(\mu_j, A)$$

Proof. This can be seen by iterating the resolvent identity (Proposition 3.3.2(c)). □

Observe that Lemma 6.4.5 is a special case of Lemma 9.2.2 for $\mu_1 = \ldots = \mu_n$.

The second ingredient in the proof of Theorem 9.2.1 is the fact that a Banach space cannot be written as a countable union of proper subspaces which are continuously embedded Banach spaces themselves. This follows from Baire's theorem if applied to so-called **operator ranges**.

Definition 9.2.3 (Operator range)**.** A vector subspace V of a Banach space X is called an **operator range in** X if there exists a complete norm $\|\cdot\|_V$ on V which makes the inclusion map from V into X continuous.

The terminology is due to the following fact: a vector subspace V of a Banach space X can be shown to be an operator range if and only if there exists a Banach space Z and a bounded linear operator $T \in \mathcal{L}(Z, X)$ with $\operatorname{rg} T = V$. This equivalence is not needed in the main text, but since it motivates the terminology we include a proof, along with several interesting facts about operator ranges, in Supplement 9.A. We do need the following properties.

Proposition 9.2.4. *Let X, Y be Banach spaces.*

(a) *If $T \in \mathcal{L}(X, Y)$ and $W \subseteq Y$ is an operator range in Y, then $T^{-1}(W)$ is an operator range in X.*

(b) (Baire's theorem for operator ranges) *Every operator range $V \subsetneq X$ is meagre.*[1] *Hence, if $X = \bigcup_{n\in\mathbb{N}} V_n$ for operator ranges V_n, then $X = V_{n_0}$ for at least one $n_0 \in \mathbb{N}$.*

Proof. (a) Let $\|\cdot\|_W$ be a norm on W that makes W complete the inclusion map from W into Y continuous. One can check that the pre-image $V := T^{-1}(W)$ endowed with the norm $\|v\|_V := \|v\|_X + \|Tv\|_W$ is complete and and obviously $V \hookrightarrow X$.

(b) Let $\|\cdot\|_V$ be a norm on V such that V is complete and the inclusion map $J\colon V \to X$ is continuous. In the proof of the open mapping theorem, one shows that if a bounded linear operator between two Banach spaces is not surjective, then the image of the unit ball under this operator is nowhere dense [Rud91, Theorem 2.11]. By applying this to J, one sees that the unit ball of $(V, \|\cdot\|_V)$ is nowhere dense in $(X, \|\cdot\|_X)$. Hence, V is meagre in X. The second assertion now follows from Baire's theorem. ☐

Proof of Theorem 9.2.1. Let $m \geq 2$ since if $m \leq 1$, there is nothing to prove. We first show that $\mathcal{R}(\lambda, A)f \succeq -u$ for each $\lambda \in (-\infty, \lambda_0) \cap \rho(A)$ and each $f \in E_+$. Indeed, given such an f and such a λ, the individual eventual positivity of $\mathcal{R}(\cdot, A)$ at λ_0 with respect to 0 allows us to recursively find numbers $\mu_1, \dots, \mu_{m-1} > \lambda$ such that $\mathcal{R}(\mu_1, A)\cdots\mathcal{R}(\mu_k, A)f \geq 0$ for all $k \in \{1, \dots, m-1\}$. We set $S := \mathcal{R}(\lambda, A)\prod_{j=1}^{m-1}(\mu_j - \lambda)\prod_{j=1}^{n}\mathcal{R}(\mu_j, A)$. Then

$$\mathcal{R}(\lambda, A)f \geq Sf \succeq -u,$$

where the first estimate follows from the expansion of $\mathcal{R}(\lambda, A)$ in Lemma 9.2.2 and the fact that $\lambda < \lambda_0 < \mu_j$ for all j, while the second estimate holds since $\operatorname{rg} S \subseteq \operatorname{dom}(A^m) \subseteq E_u$ and since Sf is real.

Now consider the countable set $C := \rho(A) \cap \mathbb{Q} \cap (-\infty, \lambda_0)$. For each $f \in E_+$ one has

$$-u \preceq \mathcal{R}(\lambda, A)f \leq 0, \qquad \text{so} \qquad \mathcal{R}(\lambda, A)f \in E_u$$

for all $\lambda \in C$ in an f-dependent left neighbourhood of λ_0. The first estimate was shown in the first part of the proof and the second one follows from the eventual negativity assumption. Since E_+ spans E, we conclude that for each $f \in E$ there exists a $\lambda \in C$

[1] Recall that a subset of a metric (or topological) space is called **meagre** if it is the union of countably many nowhere dense sets. A set is called **nowhere dense** if its closure has empty interior.

such that $\mathcal{R}(\lambda, A)f \in E_u$. Hence, $E = \bigcup_{\lambda \in C} \mathcal{R}(\lambda, A)^{-1}(E_u)$. Since E_u is an operator range (Proposition 7.1.6), so is its pre-image $\mathcal{R}(\lambda, A)^{-1}(E_u)$ for each λ (Proposition 9.2.4(a)). As C is countable, Baire's theorem for operator ranges (Proposition 9.2.4(b)) implies that $E = \mathcal{R}(\lambda, A)^{-1}(E_u)$ for some $\lambda \in C$. Consequently, $\operatorname{dom}(A) = \mathcal{R}(\lambda, A)E \subseteq E_u$. □

If an operator A on a Banach lattice E satisfies $\operatorname{dom}(A^m) \subseteq E_u$ for an integer $m \in \mathbb{N}$ and a vector $u \in E_+$, one can use Theorem 9.2.1 improve the characterisation of individual eventual positivity and negativity from Section 7.3. One can now incorporate the domination assumption $\operatorname{dom}(A) \subseteq E_u$ that occurs in Theorem 7.3.6 and in Corollary 7.3.7 – which is much stronger than $\operatorname{dom}(A^m) \subseteq E_u$ – into the equivalent conditions:

Corollary 9.2.5. *Let $A\colon E \supseteq \operatorname{dom}(A) \to E$ be a closed, densely defined, and real operator on a complex Banach lattice E and let $\lambda_0 \in \mathbb{R}$ be a pole of the resolvent $\mathcal{R}(\cdot, A)$. Let $u \in E_+$ be such that $\operatorname{dom}(A^m) \subseteq E_u$ for some $m \in \mathbb{N}$. Then the following are equivalent.*

(i) *$\mathcal{R}(\cdot, A)$ is individually eventually positive and negative with respect to u at λ.*

(ii) *The spectral projection P associated to λ_0 satisfies $Pf \succeq u$ whenever $0 \lneq f \in E$, and* $\operatorname{dom}(A) \subseteq E_u$.

(iii) *The eigenspace $\ker(\lambda_0 - A)$ is spanned by a vector $v \succeq u$, the dual eigenspace $\ker(\lambda_0 - A')$ contains a strictly positive functional ψ, and* $\operatorname{dom}(A) \subseteq E_u$.

Proof. The implications (ii) ⇔ (iii) ⇒ (i) hold by Theorem 7.3.6 and Corollary 7.3.7.

"(i) ⇒ (ii)": By Theorem 7.3.6 we get $Pf \succeq u$ for all $0 \lneq f \in E_+$ and from Theorem 9.2.1 one obtains $\operatorname{dom}(A) \subseteq E_u$. □

9.3 Eventual negativity for the Dirichlet and Neumann Laplacians

In many previous examples, we proved partial results about (eventual) positivity and eventual negativity of $\mathcal{R}(\cdot, \Delta_{\mathrm{Dir}})$, where Δ_{Dir} denotes the Dirichlet Laplacian on $L^2(\Omega)$ for a bounded domain $\Omega \subseteq \mathbb{R}^n$. We are now in a position to give a complete description of eventual negativity of the resolvent of Δ_{Dir} on sufficiently smooth bounded domains. It is instructive to compare the following characterisation with Example 6.4.6, which was not sharp at all.

Example 9.3.1 (Eventual negativity for the Dirichlet Laplacian)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open, bounded, and connected. For simplicity, assume that Ω has C^∞ boundary. Consider the quasi-interior point δ of $L^2(\Omega)_+$ given by $\delta := \operatorname{dist}(\cdot, \partial\Omega)$. The following are equivalent:

(i) $\mathcal{R}(\cdot, \Delta_{\mathrm{Dir}})$ is individually eventually negative with respect to 0 at $\mathrm{s}(\Delta_{\mathrm{Dir}})$.

(ii) $\mathcal{R}(\cdot, \Delta_{\mathrm{Dir}})$ is individually eventually negative with respect to δ at $\mathrm{s}(\Delta_{\mathrm{Dir}})$.

(iii) The spatial dimension is $n = 1$.

Moreover, $\mathcal{R}(\cdot, \Delta_{\mathrm{Dir}})$ is not uniformly eventually negative with respect to 0 at $\mathrm{s}(\Delta_{\mathrm{Dir}})$.

Proof. "(iii) ⇒ (ii)": If $n = 1$, then $\Omega = (a, b)$ is a bounded open interval.

Without loss of generality, assume that $(a, b) = (0, \pi)$. We computed in Example 6.3.2 that $\sigma(\Delta_{\mathrm{Dir}}) = \{-k^2 : k \in \mathbb{N}\}$ and the eigenspace corresponding to $\mathrm{s}(\Delta_{\mathrm{Dir}}) = -1$ is spanned by $u = \sin(\cdot) \geq 2\pi^{-1}\delta$.[2] As Δ_{Dir} is associated to a symmetric form, it is self-adjoint (Proposition 8.4.3), so the dual eigenspace is also spanned by u.

In order to conclude the individual eventual negativity, it remains to show $\operatorname{dom}(\Delta_{\mathrm{Dir}}) \subseteq L^2(0, \pi)_\delta$ due to Corollary 7.3.7. Let $f \in \operatorname{dom}(\Delta_{\mathrm{Dir}})$. Then $f \in H^2(0, \pi) \subseteq \mathrm{C}^1([0, \pi])$ by the Sobolev embedding theorem (Theorem 5.3.7(b)). Also, $f \in H_0^1(0, \pi)$ which implies that $f(0) = f(\pi) = 0$ by Exercise 6.3. Thus, if $x \in [0, \pi/2]$, then

$$\left|f(x)\right| = \left|\int_0^x f'(t)\,\mathrm{d}t\right| \leq \left\|f'\right\|_\infty x$$

by the fundamental theorem of calculus. Likewise, if $x \in [\pi/2, \pi]$, then we obtain $\left|f(x)\right| \leq \left\|f'\right\|_\infty (\pi - x)$. In either case, $\left|f\right| \preceq \delta$, i.e. $f \in L^2(0, \pi)_\delta$.

"(ii) ⇒ (i)": This implication is clear.

"(i) ⇒ (iii)": Since Ω has smooth boundary, it holds that $\operatorname{dom}\left(\Delta_{\mathrm{Dir}}^m\right) \subseteq L^2(\Omega)_\delta$ for sufficiently large m by Example 7.2.4(a). Moreover, $\mathcal{R}(\lambda, \Delta_{\mathrm{Dir}}) \geq 0$ for all $\lambda > \mathrm{s}(\Delta_{\mathrm{Dir}})$ by Example 5.4.2. Hence, Theorem 9.2.1 implies that $\operatorname{dom}(\Delta_{\mathrm{Dir}}) \subseteq L^2(\Omega)_\delta$.

Suppose that $n \geq 2$. We may assume that $0 \in \partial\Omega$. We construct $u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that u/δ is not bounded in any neighbourhood of 0 in order to contradict that $\operatorname{dom}(\Delta_{\mathrm{Dir}}) \subseteq L^2(\Omega)_\delta$. We only show the existence of such a function u in the situation when Ω is the half-space $\{x \in \mathbb{R}^n : x_n > 0\}$; this is of course an unbounded set, but we explain in Remark 9.3.2 below how the existence of u on bounded domains Ω with C^∞-boundary can be derived from the existence on the half-space.

We consider the function $\tilde{u} \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ given by $\tilde{u}(x) := x_n h(\|x\|)$, where $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^n$, $x = (\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times R$, and $h \in \mathrm{C}((0, \infty); \mathbb{R})$ is given by $h(t) := \left|\log t\right|^\alpha$ for an arbitrary but fixed $0 < \alpha < \frac{1}{2}$. Obviously, $|\tilde{u}| \not\preceq \delta$ because of the singularity at 0. Note that h is C^∞ on $(0, 1)$, so $\tilde{u}$ is C^∞ on $B_{<1}(0) \setminus \{0\}$. One can verify by direct computation that

$$\int_0^{1/2} \left|h'(r)\right|^2 r^{n-1}\,\mathrm{d}r < \infty \qquad \text{and} \qquad \int_0^{1/2} \left|h''(r)\right|^2 r^{n+1}\,\mathrm{d}r < \infty.$$

Using these one can derive by elementary yet tedious calculations that the restriction of u to the pointed disk $B_{<1/2}(0) \setminus \{0\}$ is in $H^2\left(B_{<1/2}(0) \setminus \{0\}\right)$; the interested reader can find more details in Proposition 9.B.2 in the supplementary Section 9.B.

Finally, we choose $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n)$ that is constantly 1 in a neighbourhood of 0 and vanishes outside $B_{<1/4}(0)$. Then $u := (\tilde{u}\varphi)|_\Omega \in \mathrm{C}^\infty(\Omega) \cap H^2(\Omega) \cap H_0^1(\Omega)$, but $|u| \not\preceq \delta$.

[2] Draw a picture to see this inequality!

No uniform eventual negativity: By the equivalence shown above, we only need to consider the case $n = 1$. In this case, we can again assume without loss of generality, that $\Omega = (0,1)$. Let $w(x) = x(1-x)$. One can check (Exercise 9.2) that $\mathcal{R}(0,\Delta_{\text{Dir}})$ is the operator T from Example 8.2.5 and hence, $\mathcal{R}(0,\Delta_{\text{Dir}}) \not\preceq w \otimes w$ according to that example. Hence, $\mathcal{R}(\cdot,\Delta_{\text{Dir}})$ does not satisfy the conclusion of Theorem 9.1.1 and thus, it does not satisfy condition (3) in the theorem (because it satisfies the other conditions). But we know from Example 5.4.2 that $\mathcal{R}(\lambda,\Delta_{\text{Dir}}) \geq 0$ for all $\lambda > \text{s}(\Delta_{\text{Dir}})$, so it is the uniform eventual negativity that does not hold. □

Remark 9.3.2. In Example 9.3.1, we constructed a function $\tilde{u} \in H^2(\Omega) \cap H^1_0(\Omega)$ on the open half-space $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$. Evidently this Ω is not a bounded set, so the reader should rightly question why this is appropriate. Actually, if $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is open and bounded with C^k boundary for some $k \in \mathbb{N} \cup \{\infty\}$, many questions of regularity of PDE solutions can be reduced to the study of functions on the half-space. (Indeed, Theorem 5.3.2 on elliptic regularity for the Dirichlet Laplacian is first established on a half-space). The general case is obtained by the following technical procedure.

Since $\partial\Omega$ is of class C^k, one can show that for every $x_0 \in \partial\Omega$, there is an open set $U \subseteq \mathbb{R}^n$ (called a coordinate neighbourhood) with $U \cap \partial\Omega \ni x_0$ and a C^k diffeomorphism $\Phi\colon U \cap \Omega \to G$ with $\Phi(x_0) = 0$ and $G = B_{<r}(0) \cap \{x_n > 0\}$ for some $r > 0$ (i.e. G is a set of the form Ω_R used in Example 9.3.1). Hence the map Φ "flattens the boundary" near x_0 – see, for instance, [Eva10, Appendix C.1] for a precise formulation of this technique. Fortunately, it turns out that Sobolev spaces are well-behaved under C^k change of coordinates, as shown in [AF03, Theorem 3.41]. Thus, many technical constructions can be carried out 'locally' in the much easier setting of the set G, and then transferred via diffeomorphism back onto Ω. Finally, since $\partial\Omega$ is compact, it is covered by finitely many coordinate neighbourhoods U, and global results on $\overline{\Omega}$ are then obtained by patching things up via a partition of unity.

Let us discuss a different version of the Laplace operator now, which we have already seen in Example 8.5.2 in the one-dimensional case: the Neumann Laplacian.

Example 9.3.3 (The Neumann Laplacian on bounded domains)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open, bounded, and connected. We assume, for the sake of simplicity, that Ω has C^∞ boundary. Consider the sesquilinear form $\mathfrak{a}\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ given by

$$\mathfrak{a}(v,w) := (\nabla v \mid \nabla w)_{L^2} \qquad \text{for all } v,w \in H^1(\Omega).$$

The associated operator $\Delta_{\text{Neu}}\colon L^2(\Omega) \supseteq \text{dom}(\Delta_{\text{Neu}}) \to L^2(\Omega)$ – called the **Neumann Laplacian** or the **Neumann Laplace operator**[3] – has the following properties.

(a) Δ_{Neu} is closed, densely defined, and self-adjoint.

[3] One can prove that functions $u \in \text{dom}(\Delta_{\text{Neu}})$ have, in an appropriate sense, vanishing normal derivative on $\partial\Omega$. We refrain from discussing this here in detail and instead refer to the one-dimensional case for intuition, where this is a special case of Exercise 5.6(a).

(b) Δ_{Neu} has compact resolvent and its spectral bound is $\operatorname{s}(\Delta_{\text{Neu}}) = 0$. The corresponding eigenspace $\ker \Delta_{\text{Neu}}$ is spanned by the constant function $\mathbb{1}$.

(c) One has $\mathcal{R}(\cdot, \Delta_{\text{Neu}}) \geq 0$ for all $\lambda \in (0, \infty)$.

(d) For every $k \in \mathbb{N}$ one has the elliptic regularity result $\operatorname{dom}\left(\Delta_{\text{Neu}}^k\right) \subseteq H^{2k}(\Omega)$.

Proof. (a) By the Cauchy–Schwarz inequality, one sees that

$$|\mathfrak{a}(v, w)| \leq \|\nabla v\|_{L^2} \|\nabla w\|_{L^2} \leq \|v\|_{H^1} \|w\|_{H^1}$$

and for each $\mu \in (0, \infty)$,

$$\operatorname{Re} \mathfrak{a}(v, v) + \mu \|v\|_{L^2}^2 \geq (\mu \wedge 1)\left(\|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2\right) = (\mu \wedge 1) \|v\|_{H^1}^2$$

for all $v, w \in H^1(\Omega)$. Thus $\mathfrak{a}$ is a bounded sesquilinear form satisfying an ellipticity estimate of the form (5.1.1). So, A is closed and densely defined by Theorem 5.1.4(a). Clearly, $\mathfrak{a}$ is symmetric, so the self-adjointness follows from Proposition 8.4.3.

(b) The embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact by Theorem 6.3.6(b), and hence it follows from Proposition 6.2.10(b) that Δ_{Neu} has compact resolvent. Owing to the ellipticity estimate shown above, we conclude from Theorem 5.1.4(b) that $\operatorname{s}(A) \leq \mu$ for all $\mu > 0$ and hence $\operatorname{s}(A) \leq 0$.

One immediately sees that $\Delta_{\text{Neu}} \mathbb{1} = 0$, i.e. 0 is an eigenvalue of Δ_{Neu} and $\mathbb{1} \in \ker \Delta_{\text{Neu}}$. In particular, $\operatorname{s}(\Delta_{\text{Neu}}) = 0$. Furthermore, if $v \in \ker \Delta_{\text{Neu}}$, then $\mathfrak{a}(v, v) = 0$ and so $\nabla v = 0$ a.e. in Ω. However, we see below that $v \in \operatorname{dom}\left(A^k\right) \subseteq H^{2k}(\Omega)$ for each $k \in \mathbb{N}$, and the latter space embeds into $\mathrm{C}^1(\overline{\Omega})$ for sufficiently large $k \in \mathbb{N}$ (Theorem 5.3.4). Hence $\nabla v = 0$ in the classical sense. This implies that v is constant, since Ω is connected. We have thus shown that the $\ker \Delta_{\text{Neu}}$ is spanned by $\mathbb{1}$.

(c) Recall from Example 4.1.4(d) (Stampacchia's lemma) that $H^1(\Omega; \mathbb{R})$ is a vector sublattice of $L^2(\Omega; \mathbb{R})$, and for every $v \in H^1(\Omega; \mathbb{R})$, we have $\mathfrak{a}(v^+, v^-) = 0$. Hence the assertion follows from the Beurling–Deny criterion in Theorem 5.1.7.

(d) We do not prove this result and instead (as in Chapter 5) refer to the PDE literature, for example [Bre11, Theorem 9.26]. □

Just as for the Dirichlet Laplacian, one can also characterise in terms of the dimension when the resolvent $\mathcal{R}(\cdot, \Delta_{\text{Neu}})$ is individually eventually negative; see Exercise 9.5. Let us point out again that, in contrast to the Dirichlet Laplacian, for $\mathcal{R}(\cdot, \Delta_{\text{Neu}})$ one even gets uniform eventual negativity if $n = 1$ (Example 8.5.2).

We close this chapter with following example of a fourth-order operator on a domain in at most three dimensions. While it does not require the main results of this chapter, but this seems to be a good place for the example anyway as it builds on the Neumann Laplacian just introduced.

Example 9.3.4 (Minus the square of the Neumann Laplacian in dimensions ≤ 3)**.** In the situation of Example 9.3.3, let $n \leq 3$ and set $B := -\Delta_{\mathrm{Neu}}^2$. Then B is self-adjoint, has compact resolvent, and $\mathrm{s}(B) = 0$. The resolvent $\mathcal{R}(\cdot, B)$ is uniformly eventually positive and negative with respect to $\mathbb{1} \otimes \mathbb{1}$ at 0, but for all sufficiently large $\lambda > 0$ one has $\mathcal{R}(\lambda, B) \not\geq 0$.

Proof. In Exercise 9.4(a), it is shown (in a more abstract setting) that B is self-adjoint and that $\mathrm{s}(B) \leq 0$. Moreover it follows from $\mathbb{1} \in \ker \Delta_{\mathrm{Neu}}$ that $\mathbb{1} \in \ker B$, so $\mathrm{s}(B) = 0$. The claim that $\mathcal{R}(\lambda, A) \not\geq 0$ if $\lambda > 0$ is sufficiently large is shown in Exercise 9.4(b).

To show compactness of the resolvent of B, choose a number $\lambda \in \mathbb{R}$ such that $\pm\lambda \notin \sigma(\Delta_{\mathrm{Neu}})$. Then $-\lambda^2 - B = (\Delta_{\mathrm{Neu}} - \lambda)(\Delta_{\mathrm{Neu}} + \lambda)$, so $-\lambda^2 \in \rho(B)$ and

$$\mathcal{R}(-\lambda^2, B) = \mathcal{R}(\lambda, \Delta_{\mathrm{Neu}})\mathcal{R}(-\lambda, \Delta_{\mathrm{Neu}}). \tag{9.3.1}$$

Thus, $\mathcal{R}(-\lambda^2, B)$ is compact.

To show the uniform eventual positivity and negativity, we use Theorem 8.3.2. The spectral value $\mathrm{s}(B)$ is a pole of $\mathcal{R}(\cdot, B)$ since B has compact resolvent, so we only need to check the assumptions (1)–(3) of the theorem.

(1) We have already observed that $\mathbb{1} \in \ker B$. Conversely, assume now that $v \in \ker B$. Then $\Delta_{\mathrm{Neu}}^2 v = 0$. The self-adjointness of Δ_{Neu} thus gives

$$0 = \left(v \mid \Delta_{\mathrm{Neu}}^2\right)_{L^2} = (\Delta_{\mathrm{Neu}} v \mid \Delta_{\mathrm{Neu}} v)_{L^2} = \|\Delta_{\mathrm{Neu}} v\|_{L^2}^2,$$

so $v \in \ker \Delta_{\mathrm{Neu}}$ and thus, v is a multiple of $\mathbb{1}$ by Example 9.3.3(b).

(2) Since B is self-adjoint, it follows from (1) that $\mathbb{1} \in \ker B'$.

(3) One has $\mathrm{dom}(\Delta_{\mathrm{Neu}}) \subseteq H^2(\Omega) \subseteq \mathrm{C}(\overline{\Omega}) \subseteq L^\infty(\Omega)$, where the first inclusion is the elliptic regularity result cited in Example 9.3.3(d) (for $k = 1$), and the second inclusion follows the Sobolev embedding result from Theorem 5.3.4 since $n < 4$. So $\mathcal{R}(\lambda, \Delta_{\mathrm{Neu}})$ and $\mathcal{R}(-\lambda, \Delta_{\mathrm{Neu}})$ map $L^2(\Omega)$ into $L^\infty(\Omega) = (L^2(\Omega))_{\mathbb{1}}$. The same is true for their dual operators as the operators are self-adjoint and real. Thus, the product representation of $\mathcal{R}(-\lambda^2, B)$ in (9.3.1) and Corollary 8.2.4 imply that $\pm\mathcal{R}(-\lambda^2, B) \preceq \mathbb{1} \otimes \mathbb{1}$. □

Exercises for Chapter 9

Exercise 9.1. Let $A\colon E \supseteq \operatorname{dom}(A) \to E$ be a closed and densely defined operator with non-empty resolvent set on a Banach lattice E, let $u \in E_+$, $\varphi \in E'_+$, and $m \in \mathbb{N}$.

(a) Show that $\operatorname{dom}(A^m)$ is dense in E and that $\operatorname{dom}\big((A')^m\big)$ is weak*-dense in E'.

(b) Deduce that u is a quasi-interior point of E_+ if $\operatorname{dom}(A^m) \subseteq E_u$.

(c) Show that if $(E')_\varphi$ is weak*-dense in E', then φ is strictly positive.

(d) Deduce that φ is strictly positive if $\operatorname{dom}\big((A')^m\big)(E') \subseteq (E')_\varphi$.

Exercise 9.2. Consider the Dirichlet Laplacian $\Delta_{\mathrm{Dir}} : L^2(0,1) \supseteq \operatorname{dom}(\Delta_{\mathrm{Dir}}) \to L^2(0,1)$. In the proof of Example 9.3.1 it was claimed that $\mathcal{R}(0,\Delta_{\mathrm{Dir}}) = T$, where $T\colon L^2(0,1) \to L^2(0,1)$ is the operator from Example 8.2.5, i.e. it is given by

$$Tf(x) = \int_0^1 G(x,y) f(y)\,\mathrm{d}y$$

for all $f \in L^2(0,1)$ and for $x \in (0,1)$, where $G : [0,1]^2 \to [0,\infty)$ is defined by

$$G(x,y) = x \wedge y - xy$$

for all $x, y \in [0,1]$. Prove that indeed $\mathcal{R}(0,\Delta_{\mathrm{Dir}}) = T$.

Strategy: Assume $f \in \mathrm{C}([0,1])$ first, and set $u := Tf$. Show that $u \in \operatorname{dom}(\Delta_{\mathrm{Dir}})$ and that $-\Delta_{\mathrm{Dir}} u = f$.

Exercise 9.3 (Fun with operator ranges)**.**

(a) It is a classical exercise in functional analysis to show that $\bigcup_{p\in(1,\infty)} L^p(0,1) \neq L^1(0,1)$ by explicitly constructing a function that is in $L^1(0,1)$, but not in $L^p(0,1)$ for any $p \in (1,\infty)$. Give an alternative proof for $\bigcup_{p\in(1,\infty)} L^p(0,1) \neq L^1(0,1)$ by using Proposition 9.2.4(b).

(b) Let X be a Banach space and let $V \subseteq X$ be an operator range. Let $\|\cdot\|_{V,1}$ and $\|\cdot\|_{V,2}$ be two complete norms on V that make the inclusion map $V \to X$ continuous. Show that $\|\cdot\|_{V,1}$ and $\|\cdot\|_{V,2}$ are equivalent.

Exercise 9.4 (Squares of operators via forms)**.**

(a) Let H be a complex Hilbert space and $A\colon H \supseteq \operatorname{dom}(A) \to H$ a self-adjoint operator with $\operatorname{s}(A) \le 0$. Define a sesquilinear form $\mathfrak{b}\colon \operatorname{dom}(A) \times \operatorname{dom}(A) \to \mathbb{C}$ on H by $\mathfrak{b}(v, w) := (Av \mid Aw)_H$ for all $v, w \in \operatorname{dom}(A)$.

Prove that the operator associated to $\mathfrak{b}$ is $-A^2$, that $-A^2$ is self-adjoint and that $\operatorname{s}(-A^2) \le 0$.

(b) Show the claim in Example 9.3.4: $\mathcal{R}(\lambda, -\Delta_{\text{Neu}}^2) \not\geq 0$ for all sufficiently large $\lambda > 0$.

Exercise 9.5 (Individual eventual negativity for the Neumann Laplacian)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open, bounded, and connected with C^∞ boundary. We consider the Neumann Laplacian $\Delta_{\text{Neu}}\colon L^2(\Omega) \supseteq \operatorname{dom}(\Delta_{\text{Neu}}) \to L^2(\Omega)$ from Example 9.3.3. Show that the following are equivalent:

(i) $\mathcal{R}(\cdot, \Delta_{\text{Neu}})$ is individually eventually negative with respect to 0 at $\operatorname{s}(\Delta_{\text{Neu}})$.

(ii) $\mathcal{R}(\cdot, \Delta_{\text{Neu}})$ is individually eventually negative with respect to $\mathbb{1}$ at $\operatorname{s}(\Delta_{\text{Neu}})$.

(iii) The spatial dimension satisfies $n \le 3$.

Hint for the implication (i) $\Rightarrow$ (iii): Proceed similarly as in Example 6.4.6. You may use without proof that every function $u \in H^2(\Omega)$ that vanishes in a neighbourhood of $\partial\Omega$ is in $\operatorname{dom}(\Delta_{\text{Neu}})$.

Notes for Chapter 9

Eventual positivity for higher order elliptic operators

For powers of the Neumann Laplacian Δ_{Neu} or the Dirichlet Laplacian Δ_{Dir}, eventual positivity is typically not too difficult to study since the eigenspaces of those powers are determined by the eigenspaces of Δ_{Neu} and Δ_{Dir}. We demonstrated this concretely for the square of the Neumann Laplacian on two-dimensional domains (Example 9.3.4). However, the boundary conditions that one gets by squaring those operators can be a bit idiosyncratic. For instance, for a function u to be in $\operatorname{dom}\left(\Delta_{\mathrm{Dir}}^2\right)$ one has to require, loosely speaking, that u and Δu both vanish at the boundary.

In some physical models – for instance in the so-called **clamped plate equation** – different boundary conditions occur for the **bi-Laplace operator** Δ^2: there, one requires that both the trace and the normal derivative vanish on $\partial\Omega$. Let us denote this bi-Laplace operator by Δ_{cl}^2, where "cl" stands for "clamped". If Ω is sufficiently smooth, one has $\operatorname{dom}\left(\Delta_{\mathrm{cl}}^2\right) = H^4(\Omega) \cap H_0^2(\Omega)$. Those natural – and seemingly quite harmless – boundary conditions cause a complete change in the eventual positivity behaviour. Whether the bi-Laplace operator with those boundary conditions has a positive eigenfunction for its spectral bound now depends heavily on the geometry of Ω. For instance, if Ω is a ball, an explicit form for $\mathcal{R}(0, -\Delta_{\mathrm{cl}}^2)$[4] is given by **Boggio's formula**, see e.g. [GGS10, Lemma 2.27]. From there, one sees that the resolvent is positive (and satisfies a nice lower bound). One can then invoke the spectral theory of positive operators (or of eventually positive resolvents, see e.g. Theorem 7.3.6) to obtain information on the eigenspace associated to $\mathrm{s}(-\Delta_{\mathrm{cl}}^2)$. Similar results remain true on small perturbations of balls (where "small" must be interpreted in an appropriate sense). However, on more general domains such as, for instance, annuli with small inner radius, the eigenspace associated to $\mathrm{s}(-\Delta_{\mathrm{cl}}^2)$ does not contain a positive eigenfunction; see e.g. [SS20, Section 3] and [Dal05].

A survey of results regarding positivity of resolvents and eigenfunctions for higher-order elliptic operators can be found in [Swe16]. Much more in-depth results can be found in the monograph [GGS10].

[4] We are considering the operator $-\Delta_{\mathrm{cl}}^2$ rather than Δ_{cl}^2: motivated by the Laplace operator (and, much more so, by the semigroup theory presented in the later chapters, we prefer to consider operators whose spectral bound is finite. One can indeed show that $\mathrm{s}(-\Delta_{\mathrm{cl}}^2) < 0$.).

Operator ranges

Operator ranges, defined and used in Section 9.2 and further explored in Supplement 9.A, are studied in detail in [Cro80, COS95]. In eventual positivity, they are not only used to prove the necessary criterion for individual eventual positivity and negativity from Theorem 9.2.1, but also in the analysis of irreducibility and related properties for eventually positive operator semigroups, as shown in [AG24].

Encore: if you want to know more...

9.A Properties of operator ranges

In this supplemental section we collect a few interesting properties of operator ranges. According to Definition 9.2.3 we call a vector subspace V of a Banach space X is an operator range in X if there a complete norm $\|\cdot\|_V$ on V that renders V a Banach space and makes the inclusion map from V into X continuous. The following theorem explains the choice of this terminology.

Theorem 9.A.1. *The following are equivalent for a subspace V of a Banach space X.*

(i) *V is an operator range in X.*

(ii) *There exists a Banach space Z and an operator $T \in \mathcal{L}(Z, X)$ such that $\operatorname{rg} T = V$.*

(iii) *V is the domain of a closed operator on X.*

Proof. "(i) ⇒ (ii)": This implication is obvious.

"(ii) ⇒ (iii)": Since $\operatorname{rg} T = V$, the induced operator $T_/ : Z/\ker T \to X$ is bounded and bijective onto V. In particular, V is the domain of the closed operator $T_/^{-1}$.

"(iii) ⇒ (i)": If $A\colon X \supseteq V \to X$ is a closed operator, then any graph norm on V is complete and renders the inclusion map $V \to X$ continuous, i.e., V is an operator range. □

By Theorem 9.A.1, every bounded operator maps operator ranges to operator ranges. Moreover, we known from Proposition 9.2.4(a) that inverse image of an operator range under a bounded operator is also an operator range. Some other interesting properties are collected in the following result.

Proposition 9.A.2. *Let $V_1, \dots, V_n$ be operator ranges in a Banach space X.*

(a) *The subspace $\bigcap_{k=1}^n V_k$ is an operator range.*

(b) *The subspace $\sum_{k=1}^n V_k$ is an operator range in X.*

(c) *If X is the algebraically direct sum $V_1, \dots, V_n$, which we denote by $X = \bigoplus_{k=1}^n V_k$, then each V_k is closed.*

Proof. (a) For each index k, let $\|\cdot\|_{V_k}$ be a complete norm on V_k that makes the inclusion map $V_k \to X$ continuous. Then

$$\|\cdot\|_V := \sum_{k=1}^{n} \|\cdot\|_{V_k}$$

is a complete norm on $V := \bigcap_{k=1}^{n} V_k$. Moreover, $\|f\| \le \|f\|_1 \le \|f\|_V$ for all $f \in V$.

(b) *and* (c) Consider the Banach space $W := V_1 \times \ldots \times V_n$ with the norm $\|(v_1,\ldots,v_n)\|_W := \sum_{k=1}^{n} \|v_k\|_{V_k}$ and the map $J \in \mathcal{L}(W,X)$ given by $J(v_1,\ldots,v_n) = v_1 + \ldots + v_n$. Then $\sum_{k=1}^{n} V_k = \operatorname{rg} J$, which is an operator range according to Theorem 9.A.1.

Now assume that $X = \bigoplus_{k=1}^{n} V_k$. Then J is bijective and thus a homeomorphism by the open mapping theorem. For each index k the space $V_k \subseteq X$ is the image of a closed subspace of W under J and is hence closed in X. □

It is worthwhile to note that the intersection of infinitely many operator ranges need not be an operator range, in general.

Example 9.A.3. Consider the Banach lattice c_0 of all scalar-valued sequences that converge to 0, endowed with the sup norm and with the componentwise order on its real part. Note that a vector $u \in (c_0)_+$ is a quasi-interior point if and only if $u_k > 0$ for all k. It is not difficult to check that

$$\bigcap \big\{(c_o)_u : 0 \le u \in c_0 \text{ is a quasi-interior point}\big\} = c_{00},$$

where c_{00} denotes the space of sequences with only finitely many non-zero entries. Since c_{00} has a countable Hamel basis, it is not a Banach space with respect to any norm (why?) and hence not an operator range. However, each of the principal ideals $(c_0)_u$ is an operator range according to Proposition 7.1.6.

9.B Sobolev functions with prescribed singularities

In this supplemental section, we provide details for the construction used in Example 9.3.1. The following elementary lemma can be shown by means of the fundamental theorem of calculus, Hölder's inequality, and Tonelli's theorem.

Lemma 9.B.1. *Let $R, \alpha > 0$, and $p \in [1,\infty)$. Let $f\colon (0,R] \to \mathbb{C}$ be C^1. If $\int_0^R |f'(r)|^p \, r^\alpha \, \mathrm{d}r < \infty$, then also $\int_0^R |f(r)|^p \, r^{\alpha-1} \, \mathrm{d}r < \infty$.*

Proposition 9.B.2. *Let $n \ge 2$ and endow $\mathbb{R}^n$ with the Euclidean norm $\|\cdot\| := \|\cdot\|_2$. Let $R > 0$, let $h\colon (0,R] \to \mathbb{C}$ be a C^2-function, and let $w, u\colon B_{<R}(0) \setminus \{0\} \to \mathbb{C}$ be given by*

$$w(x) = h(\|x\|) \qquad \text{and} \qquad u(x) = x_n w(x)$$

for all $x = (\tilde{x}, x_n) \in \big(\mathbb{R}^{n-1} \times R\big) \cap \big(B_{<R}(0) \setminus \{0\}\big)$.

(a) *At every $x \in B_{<R}(0) \setminus \{0\}$ the gradient and the Hessian of w are given by*

$$\nabla w(x) = \frac{h'(\|x\|)}{\|x\|} x \quad \textit{and}$$
$$\mathrm{H}w(x) = \frac{h'(\|x\|)}{\|x\|}\Big(\mathrm{id} - \frac{x}{\|x\|}\frac{x^{\mathrm{T}}}{\|x\|}\Big) + h''(\|x\|)\frac{x}{\|x\|}\frac{x^{\mathrm{T}}}{\|x\|}.$$

Hence, for all such x, and for a constant $C > 0$ that only depends on n and the choice of the norm on $\mathbb{R}^{n\times n}$,

$$\|\nabla w(x)\| = \big|h'(\|x\|)\big| \quad \textit{and}$$
$$\|\mathrm{H}w(x)\| \le C\bigg(\frac{\big|h'(\|x\|)\big|}{\|x\|} + \big|h''(\|x\|)\big|\bigg).$$

(b) *At every $x \in B_{<R}(0) \setminus \{0\}$ the gradient and the Hessian of u are given by*

$$\nabla u(x) = e_n w(x) + \nabla w(x) x_n \quad \textit{and}$$
$$\mathrm{H}u(x) = x_n \mathrm{H}w(x) + \nabla w(x) e_n^{\mathrm{T}} + e_n (\nabla w(x))^{\mathrm{T}}.$$

Hence, for all such x, and for a constant $D > 0$ that only depends on n and the choice of the norm on $\mathbb{R}^{n\times n}$,

$$\|\nabla u(x)\| \le |h(\|x\|)| + \|x\|\,\big|h'(\|x\|)\big| \quad \textit{and}$$
$$\|\mathrm{H}u(x)\| \le D\Big(\big|h'(\|x\|)\big| + \|x\|\,\big|h''(\|x\|)\big|\Big).$$

(c) *Assume that*

$$\int_0^R \big|h'(r)\big|^2 r^{n-1}\,\mathrm{d}r < \infty \qquad \textit{and} \qquad \int_0^R \big|h''(r)\big|^2 r^{n+1}\,\mathrm{d}r < \infty$$

Then $u \in H^2\big(B_{<R}(0) \setminus \{0\}\big)$.

Proof. (a) *and* (b) This can be checked by a straightforward calculation.

(c) First we note that the first condition implies $\int_0^R \big|h'(r)\big|^2 r^{n+2}\,\mathrm{d}r < \infty$ and hence,

$$\int_0^R |h(r)|^2 r^{n+1}\,\mathrm{d}r < \infty$$

by Lemma 9.B.1. By using spherical coordinates one can thus see that $u \in L^2$. Using the estimates for $\|\nabla u(x)\|$ and $\|\mathrm{H}u(x)\|$ in (b), and again spherical coordinates, one obtains that all first and second order derivatives of u are also in L^2. □

We note in passing that the formula for the Hessian $\mathrm{H}w(x)$ in Proposition 9.B.2(a) immediately gives a formula for the Laplace operator applied to radially symmetric functions: one has

$$\Delta w(x) = \mathrm{tr}\big(\mathrm{H}w(x)\big) = (n-1)\frac{h'(\|x\|)}{\|x\|} + h''(\|x\|)$$

for all $x \in B_{<R}(0) \setminus \{0\}$.

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