Chapter 8

Criteria for eventual positivity of resolvents: the uniform case

After proving sufficient conditions for individual eventual positivity (and negativity) of resolvents in the previous chapter, we now turn to the uniform case. A key assumption to get individual eventual positivity in Theorem 7.3.6 was the property $\operatorname{dom}(A) \subseteq E_u$, since it gives that the resolvent maps into the principal ideal E_u. For the uniform case, we need a stronger property of the resolvent. In Sections 8.1 and 8.2 we set the stage for this, before we turn to the main theorem in Section 8.3.

8.1 Banach lattice overture: Norms induced by functionals

To obtain criteria for uniformly eventually positive resolvents, we need a few more tools from Banach lattice theory. This section introduces a construction dual to the principal ideals discussed in Section 7.1. The following example serves as motivation.

Example 8.1.1. Let (Ω, μ) be a finite measure space and let $p, p' \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. As usual we identify $L^{p'}(\Omega, \mu)$ with the dual space $\left(L^p(\Omega, \mu)\right)'$ – note that this identification respects the order structure.

The function $\mathbb{1} \in L^{p'}(\Omega, \mu)$ acts as a strictly positive functional on the Banach lattice $L^p(\Omega, \mu)$, and for each $f \in L^p(\Omega, \mu)$ one has

$$\langle \mathbb{1}, |f| \rangle = \int_\Omega |f| \, \mathrm{d}\mu = \|f\|_{L^1}.$$

Hence $\langle \mathbb{1}, |\cdot| \rangle$ is a norm on $L^p(\Omega, \mu)$. Of course, the norm completion of $L^p(\Omega, \mu)$ with respect to this norm is the Banach lattice $L^1(\Omega, \mu)$.

Now we take this example and turn it into a general construction on Banach lattices. If E, F are Banach lattices over the same field, a linear map $J \colon E \to F$ is called a **lattice homomorphism** if $|Jx| = J|x|$ for all $x \in E$. Observe that every lattice homomorphism is positive (and hence continuous by Theorem 4.3.3). A bijective lattice homomorphism is called a **lattice isomorphism**. This is justified, since one easily checks that the inverse is also a lattice homomorphism.

Proposition 8.1.2 (AL-spaces generated by functionals)**.** *Let E be a Banach lattice, let $\varphi \in E'_+$ be a strictly positive functional, and consider the norm $\|\cdot\|_{E^\varphi} := \langle \varphi, |\cdot| \rangle$ on E. There exists, up to an isometric lattice isomorphism, precisely one Banach lattice E^φ over the same field as E with the following properties:*

(a) *As a Banach space, E^φ is the norm completion of the normed space $(E, \|\cdot\|_{E^\varphi})$.*[1]

(b) *The inclusion map $E \hookrightarrow E^\varphi$ is a lattice homomorphism.*

*The Banach lattice E^φ is called the **AL-space generated by** φ.*

The terminology for E^φ is due to the fact that the norm on this space can be readily seen to be additive on the positive cone, and Banach lattices with this property are often called **AL-spaces**. In the real case, the proposition can be checked by showing that E^φ is a Banach lattice with the order induced by $\overline{E_+}^{\|\cdot\|_{E^\varphi}}$ and the embedding $E \hookrightarrow E^\varphi$ is a lattice homomorphism; see, for instance, the beginning of Section IV.3 in [Sch74]. The complex case can then be derived from the real one.

The reason why we call the construction of E^φ dual to the construction of principal ideals is explained in Exercise 8.1.

8.2 Smoothing properties of operators

Let (Ω_1, μ_1) and (Ω_2, μ_2) be finite measure spaces and let $k \in L^\infty(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$. Then

$$\begin{aligned} T_k \colon L^1(\Omega_2, \mu_2) &\to L^\infty(\Omega_1, \mu_1) \\ f &\mapsto \int_\Omega k(\cdot, y) f(y) \, \mathrm{d}\mu_2(y) \end{aligned}$$

defines a bounded linear operator. In fact, the **Dunford-Pettis theorem** says that every $T \in \mathcal{L}(L^1(\Omega_2, \mu_2), L^\infty(\Omega_1, \mu_1))$ is of this form; see for instance [Are06, Theorem 4.1.1] from the lecture notes of the 9th Internet Seminar. If k is real-valued, then T_k is real and it follows from $|k| \le \|k\|_\infty \mathbb{1}_{\Omega_1 \times \Omega_2}$ that

$$\pm T_k \le \|k\|_\infty \mathbb{1}_{\Omega_1} \otimes \mathbb{1}_{\Omega_2};$$

where we use the notation for rank-1 operators introduced in Notation 7.3.1. Now let $p_1, p_2 \in [1, \infty]$. If a real operator $T \colon L^{p_2}(\Omega_2, \mu_2) \to L^{p_1}(\Omega_1, \mu_1)$ extends to an operator $L^1(\Omega_2, \mu_2) \to L^\infty(\Omega_1, \mu_1)$, then the extended operator has the form described above. Thus, the extension and hence T itself are dominated above and below by multiples of $\mathbb{1}_{\Omega_1} \otimes \mathbb{1}_{\Omega_2}$.

We now generalise these observations to abstract Banach lattices. To recover the situation above in the following theorem, observe that the principal ideal generated by $\mathbb{1}_{\Omega_1}$ in $L^{p_1}(\Omega_1, \mu_1)$ is $L^\infty(\Omega_1, \mu_1)$, and that AL-space generated by the functional $\mathbb{1}_{\Omega_2} \in \left(L^{p_2}(\Omega_2, \mu_2)\right)'$ equals $L^1(\Omega_2, \mu_2)$.

[1] Recall that the norm completion of a normed space is unique up to isometric Banach space isomorphism.

Theorem 8.2.1. *Let E, F be Banach lattices, $u \in F_+$, and $\varphi \in E'$ a strictly positive functional. For every real operator $T \in \mathcal{L}(E, F)$, the following are equivalent:*

(i) *T extends to a bounded linear operator $\widetilde{T} \in \mathcal{L}(E^\varphi, F_u)$.*

(ii) *There exists a constant $c \geq 0$ such that $\pm T \leq c\, u \otimes \varphi$; in short, $\pm T \preceq u \otimes \varphi$.*

(iii) *There exists a constant $c' \geq 0$ such that $|Tx| \leq c' \langle \varphi, |x| \rangle\, u$ for all $x \in E$.*

If any of the above assertions hold, then $c = c' = \|\widetilde{T}\|_{F_u \leftarrow E^\varphi}$.

Proof. Note that the inclusions $j : E \hookrightarrow E^\varphi$ and $k : F_u \hookrightarrow F$ are lattice homomorphisms.

"(i) ⇔ (iii)": If (i) holds, then Diagram (8.2.1) commutes and for every $x \in E$,[2]

$$\|\widetilde{T} jx\|_{F_u} \leq c' \|jx\|_{E^\varphi} = c' \langle \varphi, |jx| \rangle = c' \langle \varphi, |f| \rangle;$$

where $c' = \|\widetilde{T}\|_{F_u \leftarrow E^\varphi}$. In turn, $|\widetilde{T} jx| \leq c' \langle \varphi, |x| \rangle\, u$. Consequently

$$|Tx| = |k\widetilde{T} jx| = |\widetilde{T} jx| \leq c' \langle \varphi, |x| \rangle\, u$$

which is (iii).

$$\begin{array}{ccc} E^\varphi & \xrightarrow{\ \widetilde{T}\ } & F_u \\ {\scriptstyle j}\uparrow & & \downarrow{\scriptstyle k} \\ E & \xrightarrow[\ T\ ]{} & F \end{array} \tag{8.2.1}$$

Conversely, if (iii) holds, then from the definitions of the norms on F_u and E^φ, $\|\widetilde{T} x\|_{F_u} \leq c' \|x\|_{E^\varphi}$ for all $x \in E$. By density of E in $(E^\varphi, \|\cdot\|_{E^\varphi})$, assertion (i) follows.

"(ii) ⇔ (iii)": The inequality in (iii) implies $-c\langle \varphi, x \rangle u \leq -|Tx| \leq Tx \leq |Tx| \leq c \langle \varphi, x \rangle u$ for all $x \in E_+$, which immediately yields (ii).

Conversely, assume that (ii) holds. Then for every $x \in E_\mathbb{R}$, we have

$$|Tx^+| \leq c \langle \varphi, x^+ \rangle u \quad \text{and} \quad |Tx^-| \leq c \langle \varphi, x^- \rangle u.$$

Thus $|Tx| \leq |Tx^+| + |Tx^-| \leq c \langle \varphi, |x| \rangle u$. For each $z \in E$ and $\theta \in [0, 2\pi]$, this gives

$$\left|\mathrm{Re}(e^{i\theta} Tz)\right| = \left|T\,\mathrm{Re}(e^{i\theta} z)\right| \leq c \left\langle \varphi, \left|\mathrm{Re}(e^{i\theta} z)\right| \right\rangle u;$$

the first equality uses that T is a real operator. Recalling our construction of the complex modulus function (Theorem 4.2.4), we deduce

$$\begin{aligned} |Tz| = \frac{1}{4} \int_0^{2\pi} \left|\mathrm{Re}(e^{i\theta} Tz)\right| \, \mathrm{d}\theta &\leq \frac{1}{4} \int_0^{2\pi} c \left\langle \varphi, \left|\mathrm{Re}(e^{i\theta} z)\right| \right\rangle u \, \mathrm{d}\theta \\ &= c \left\langle \varphi, \frac{1}{4} \int_0^{2\pi} \left|\mathrm{Re}(e^{i\theta} z)\right| \, \mathrm{d}\theta \right\rangle u = c \langle \varphi, |z| \rangle u, \end{aligned}$$

and thus the proof is complete. □

[2] In order to keep the notation reasonable, we do not show explicitly in which spaces the various moduli are taken.

Condition (i) in Theorem 8.2.1 easily gives the following consequence.

Corollary 8.2.2. *Let E be a Banach lattice, let $u \in E_+$, and let $\varphi \in E'$ be a strictly positive functional. If two real operators $T_1, T_2 \in \mathcal{L}(E)$ satisfy the equivalent assertions of Theorem 8.2.1 with $E = F$, then so do $\alpha_1 T_1 + \alpha_2 T_2$ and $T_1 S T_2$ for all $\alpha_1, \alpha_2 \in \mathbb{R}$ and for all real operators $S \in \mathcal{L}(E)$.*

The equivalent assertions in Theorem 8.2.1 are actually closely related to assumptions of the type $\operatorname{dom}(A) \subseteq E_u$ that occurred in Chapter 7.1. The following result makes the connection more explicit.

Proposition 8.2.3. *Let E, F be Banach lattices, let $\varphi \in E'$ be a strictly positive functional, and let $T \in \mathcal{L}(E, F)$. The following assertions are equivalent:*

(i) *T extends to a bounded linear operator $\widetilde{T}\colon E^\varphi \to F$.*

(ii) *The range of $T'\colon F' \to E'$ is contained in the principal ideal $(E')_\varphi$.*

In this case,

$$\|\widetilde{T}\|_{F \leftarrow E^\varphi} = \|T'\|_{(E')_\varphi \leftarrow F'}.$$

Proof. "(i) ⇒ (ii)": For all $\psi \in F'$ with $\|\psi\|_{F'} \le 1$ and all $y \in E$, we have

$$|\langle \psi, Ty \rangle| \le \|Ty\|_F \le \|\widetilde{T}\|_{F \leftarrow E^\varphi} \langle \varphi, |y| \rangle.$$

Now let $x \in E_+$ be arbitrary. The Riesz-Kantorovich formula (Theorem 4.4.2) implies

$$\begin{aligned}\langle |T'\psi|, x \rangle = \sup_{|y| \le x} |\langle T'\psi, y \rangle| &= \sup_{|y| \le x} |\langle \psi, Ty \rangle| \\ &\le \|\widetilde{T}\|_{F \leftarrow E^\varphi} \sup_{|y| \le x} \langle \varphi, |y| \rangle = \|\widetilde{T}\|_{F \leftarrow E^\varphi} \langle \varphi, x \rangle.\end{aligned}$$

This proves that $|T'\psi| \le \|\widetilde{T}\|_{F \leftarrow E^\varphi} \|\psi\|_{F'} \varphi$ for all $\psi \in F'$, and therefore $\operatorname{rg} T' \subseteq (E')_\varphi$ with $\|T'\|_{(E')_\varphi \leftarrow F'} \le \|\widetilde{T}\|_{F \leftarrow E^\varphi}$.

"(ii) ⇒ (i)": The closed graph theorem implies that $T'\colon F' \to (E')_\varphi$ is bounded. It follows by the definition of gauge norm that $|T'\psi| \le \|T'\psi\|_{(E')_\varphi} \varphi$ for all $\psi \in F'$. Hence

$$|\langle \psi, Tx \rangle| = |\langle T'\psi, x \rangle| \le \langle |T'\psi|, |x| \rangle \le \|T'\psi\|_{(E')_\varphi} \langle \varphi, |x| \rangle = \|T'\psi\|_{(E')_\varphi} \|x\|_{E^\varphi}$$

for all $x \in E$ and $\psi' \in F'$, where the first inequality is a direct consequence of the Riesz-Kantorovich formula. After taking the supremum over all $\psi \in F'$ with $\|\psi\|_{F'} \le 1$, we find

$$\|Tx\|_F \le \|T'\|_{(E')_\varphi \leftarrow F'} \|x\|_{E^\varphi}$$

for all $x \in E$. Since by definition E is dense in E^φ with respect to $\|\cdot\|_{E^\varphi}$, it follows that T extends to a bounded linear operator $\widetilde{T}\colon E^\varphi \to F$ with $\|\widetilde{T}\|_{F \leftarrow E^\varphi} \le \|T'\|_{(E')_\varphi \leftarrow F'}$. □

Corollary 8.2.4. *Let E be a Banach lattice, let $u \in E_+$ and let $\varphi \in E'_+$ be a strictly positive functional. If $T_1, T_2, S \in \mathcal{L}(E)$ are real operators and satisfy*

$$\operatorname{rg} T_1 \subseteq E_u \qquad \textit{and} \qquad \operatorname{rg} T_2' \subseteq (E')_\varphi,$$

then $T_1 S T_2$ satisfies the equivalent assertions of Theorem 8.2.1.

Proof. The closed graph theorem implies that $T_1 \in \mathcal{L}(E, E_u)$, while Proposition 8.2.3 shows that T_2 extends to an operator $\widetilde{T}_2 \in \mathcal{L}(E^\varphi, E)$. Hence $T_1 S T_2 \colon E \to E$ extends to the bounded linear operator $T_1 S \widetilde{T}_2 \colon E^\varphi \to E_u$, i.e. assertion (i) of Theorem 8.2.1 is fulfilled. □

Example 8.2.5. Consider the Banach lattice $E = L^2(0,1)$, and identify E' with E. Define a continuous function $G\colon [0,1]^2 \to [0,\infty)$ by $G(x,y) := x \wedge y - xy$. Let $T \in \mathcal{L}(E)$ be given by

$$Tf = \int_0^1 G(\cdot, y) f(y) \,\mathrm{d}y$$

for all $f \in E$. Consider the quasi-interior point $u \in L^2(0,1)_+$ given by $u(x) = x(1-x)$. Then T has the following properties:

(a) $\operatorname{rg} T \subseteq E_u$ and $\operatorname{rg} T' \subseteq E_u$. Thus, $T^2 \preceq u \otimes u$ according to Corollary 8.2.4.

(b) However, $T \not\preceq u \otimes u$.

Proof. (a) Since the kernel is symmetric (i.e. $G(x,y) = G(y,x)$), it suffices to show $\operatorname{rg} T \subseteq E_u$. For $x, y \in (0,1)$,

$$0 \le u(x)^{-1} G(x,y) = \begin{cases} x^{-1} y, & \text{if } y \le x, \\ (1-x)^{-1}(1-y), & \text{if } y \ge x, \end{cases}$$

hence $u(x)^{-1} G(x, \cdot) \le 1$ for all $x \in (0,1)$. Therefore, for each $f \in E$ we have

$$\left|(Tf)(x)\right| \le u(x) \int_0^1 \left|f(y)\right| \,\mathrm{d}y \le u(x) \left\|f\right\|_2,$$

which proves $Tf \in E_u$.

(b) For $\delta > 0$ we have $\frac{G(\delta,\delta)}{u(\delta)u(\delta)} = \frac{1}{\delta(1-\delta)} \to \infty$ as $\delta \downarrow 0$, in turn $T \not\preceq u \otimes u$. □

8.3 A sufficient condition for uniform eventual positivity

Our key assumption to get a sufficient condition for uniform eventual positivity (or negativity) of resolvents is that the resolvent satisfies a kernel estimate as described for general operators in Theorem 8.2.1 above. Let us first note that the validity of such an estimate does not depend on the point that one considers within the resolvent set.

Proposition 8.3.1. *Let $A\colon E \supseteq \operatorname{dom}(A) \to E$ be a closed, densely defined, and real operator on a complex Banach lattice E. Let $u \in E_+$ and let $\varphi \in E'_+$. If there exists a number $\lambda \in \rho(A) \cap \mathbb{R}$ such that $\pm\mathcal{R}(\lambda, A) \preceq u \otimes \varphi$, then the same is true for all $\lambda \in \rho(A) \cap \mathbb{R}$.*

Proof. If the given estimate is true for at least one number $\lambda \in \rho(A) \cap \mathbb{R}$, then $\operatorname{dom}(A) \subseteq E_u$ and $\operatorname{dom}\left(A'\right) \subseteq (E')_\varphi$. For all other $\tilde{\lambda} \in \rho(A) \cap \mathbb{R}$, one has the resolvent identity

$$\mathcal{R}(\tilde{\lambda}, A) = \mathcal{R}(\lambda, A) + (\lambda - \tilde{\lambda})\mathcal{R}(\tilde{\lambda}, A)\mathcal{R}(\lambda, A),$$

which gives the claim, since $\pm\mathcal{R}(\tilde{\lambda}, A)\mathcal{R}(\lambda, A) \preceq u \otimes \varphi$ by Corollary 8.2.4. □

The main result of this chapter is the following theorem which contains the sufficient conditions for uniform eventual positivity (and negativity) promised in the introduction.

Theorem 8.3.2. *Let $A\colon E \supseteq \operatorname{dom}(A) \to E$ be a closed, densely defined, and real operator on a complex Banach lattice E. Let $\lambda \in \sigma(A) \cap \mathbb{R}$ be a pole of the resolvent $\mathcal{R}(\cdot, A)$ and assume the following properties:*

1. *The eigenspace $\ker(\lambda - A)$ is spanned by a quasi-interior point $u \in E_+$.*
2. *The dual eigenspace $\ker(\lambda - A')$ contains a strictly positive functional φ.*
3. *There exists a number $\lambda_1 \in \rho(A) \cap \mathbb{R}$ such that $\pm\mathcal{R}(\lambda_1, A) \preceq u \otimes \varphi$.*

Then $\mathcal{R}(\cdot, A)$ is uniformly eventually positive with respect to $u \otimes \varphi$ at λ and uniformly eventually negative with respect to $u \otimes \varphi$ at λ.

Proof. Without loss of generality, we assume that $\lambda = 0$ and $\langle \varphi, u \rangle = 1$.

Step 1: Since φ is a strictly positive eigenvector of A' and $u \in \ker A$ is a quasi-interior point of E_+, we know from Theorem 7.3.6 that 0 is a first order pole and the associated spectral projection P satisfies $Pf \succeq u$ for all $0 \lneq u \in E$. Thus P is a rank-one projection with $Pu = u$ and $P'\varphi = \varphi$. Consequently, $P = u \otimes \varphi$.

Step 2: Let $\mu \in \rho(A) \cap \mathbb{R}$. By the finite expansion of the resolvent (Lemma 6.4.5 for $n = 2$),

$$\mathcal{R}(\mu, A) = \mathcal{R}(\lambda_1, A) + (\lambda_1 - \mu)\mathcal{R}(\lambda_1, A)^2 + R_\mu,$$

where $R_\mu := (\lambda_1 - \mu)^2\mathcal{R}(\lambda_1, A)\mathcal{R}(\mu, A)\mathcal{R}(\lambda_1, A)$. By assumption, $\mathcal{R}(\lambda_1, A)$ satisfies the equivalent assertions of Theorem 8.2.1, hence so does $\mathcal{R}(\mu, A)$ by Corollary 8.2.2.

Step 3: Using $\lambda_1\mathcal{R}(\lambda_1, A)P = P$, we can write

$$\mu R_\mu - P = \mathcal{R}(\lambda_1, A)\big((\lambda_1 - \mu)^2\mu\mathcal{R}(\mu, A) - \lambda_1^2 P\big)\mathcal{R}(\lambda_1, A).$$

Therefore, $\mu R_\mu - P$ also satisfies the equivalent assertions of Theorem 8.2.1 by Corollary 8.2.2. Combining this with the fact $\mu\mathcal{R}(\mu, A) \to P$ in $\mathcal{L}(E)$ as $\mu \to 0$, it follows that $\mu R_\mu \to P$ in $\mathcal{L}(E^\varphi, E_u)$ as $\mu \to 0$.

Step 4 : Steps 2 and 3 together give that even $\mu\mathcal{R}(\mu, A) \to P$ in $\mathcal{L}(E^\varphi, E_u)$ as $\mu \to 0$. Since $P = u \otimes \varphi$ by Step 1, it follows that $\mu\mathcal{R}(\mu, A) \succeq u \otimes \varphi$ for all μ in a neighbourhood of 0. Thus, $\mathcal{R}(\cdot, A)$ is uniformly eventually positive with respect to $u \otimes \varphi$ at 0. □

We now discuss three examples for uniform eventual positivity and negativity, each of them on a bounded interval. For the first one, everything can be computed explicitly, for the other two we use Theorem 8.3.2.

Example 8.3.3 (A first order differential operator)**.** Let $p \in [1,\infty)$. Consider the Banach lattice $E = L^p(0,1)$ and its dual space $E' = L^{p'}(0,1)$, where $p' \in (1,\infty]$ satisfies $\frac{1}{p} + \frac{1}{p'} = 1$. The closed operator A_0 on $L^p(0,1)$ given by

$$\begin{aligned} \operatorname{dom}(A_0) &:= \{f \in W^{1,p}(0,1) : f(0) = f(1)\} \\ A_0 f &:= f' \end{aligned}$$

has the following properties.

(a) A_0 has compact resolvent and its spectrum is $\sigma(A_0) = 2\pi\mathrm{i}\mathbb{Z}$.

(b) The resolvent of A_0 satisfies

$$\mathcal{R}(\mu, A_0) \preceq -\mathbb{1}\otimes\mathbb{1} \text{ if } \mu \in (-\infty, 0) \qquad \text{and} \qquad \mathcal{R}(\mu, A_0) \succeq \mathbb{1}\otimes\mathbb{1} \text{ if } \mu \in (0,\infty).$$

Proof. (a) One can verify that for each $\mu \in \mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$, the integral operator with kernel

$$K_\mu(x,y) = e^{\mu(x-y)}\left(-\mathbb{1}_{[y\le x]} + \frac{1}{1-e^{-\mu}}\right). \tag{8.3.1}$$

is inverse to $\mu - A_0$. Hence, $\mu \in \rho(A_0)$ and $\mathcal{R}(\mu, A_0)$ is the integral operator with kernel K_μ. On the other hand, for $\mu \in 2\pi\mathrm{i}\mathbb{Z}$ the function $v \in L^p(0,1)$ given by $v(x) = e^{\mu x}$ is clearly in $\operatorname{dom}(A_0)$ and satisfies $A_0 v = \mu v$, so indeed $\sigma(A_0) = 2\pi\mathrm{i}\mathbb{Z}$. The compactness of the resolvent follows, for instance, from the compactness of the embedding $W^{1,p}(0,1) \hookrightarrow L^p(0,1)$ (Theorem 6.3.1 and Exercise 6.5).

(b) Let $\mu \in \mathbb{R}\setminus\{0\}$. We use that the resolvent $\mathcal{R}(\mu, A_0)$ has the integral kernel given by formula (8.3.1). If $\mu < 0$ the summand $1/(1-e^{-\mu})$, and hence $K_\mu(x,y)$, is strictly negative. By continuity, there exists $c := -\max_{x,y\in[0,1]} K_\mu(x,y) > 0$ such that

$$(\mathcal{R}(\mu, A_0)f)(x) = \int_0^1 K_\mu(x,y) f(y)\,\mathrm{d}y \le -c\int_0^1 f(y)\,\mathrm{d}y = -c\big((\mathbb{1}\otimes\mathbb{1})f\big)(x)$$

for all $0 \le f \in E$. The proof in the case $\mu > 0$ is similar. □

Example 8.3.4 (A third order differential operator)**.** On the Banach lattice $E = L^2(0,1)$, consider the closed operator

$$\begin{aligned} \operatorname{dom}(A) &:= \{f \in H^3(0,1) : f^{(k)}(0) = f^{(k)}(1)\ \forall k = 0,1,2\} \\ Af &:= f''', \end{aligned} \tag{8.3.2}$$

and identify E' with E. Then $0 \in \sigma(A)$ is an isolated spectral value of A and $\mathcal{R}(\cdot, A)$ is uniformly eventually positive with respect to $\mathbb{1}\otimes\mathbb{1}$ at 0.

Proof. If A_0 denotes the operator from Example 8.3.3 for $p = 2$, one has $A = A_0^3$. Since $\sigma(A_0) \subseteq \mathrm{i}\mathbb{R}$, this implies that $\rho(A) \neq \emptyset$ and that A has compact resolvent. We verify the assumptions of Theorem 8.3.2.

(1) *and* (2) Observe that $\mathbb{1} \in \ker A$ and all eigenfunctions of A corresponding to the eigenvalue 0 are polynomials of degree at most 2. The periodic boundary conditions imposed in (8.3.2) ensure that they are constant. In other words, $\ker A$ is spanned by $\mathbb{1}$. Since $A' = -A$, the dual eigenspace $\ker A'$ is also spanned by $\mathbb{1}$.

(3) On E, consider the closed operator

$$\begin{aligned} \operatorname{dom}(B) &:= \{u \in H^2(0,1) : u^{(k)}(0) = u^{(k)}(1)\ \forall\, k = 0,1\} \\ Bf &:= f + f' + f'' \end{aligned}$$

Then $1 - A = (1 - A_0)B$ and $1 \in \rho(A) \cap \rho(A_0)$; where A_0 denotes the operator in Example 8.3.3. Therefore $0 \in \rho(B)$ and

$$\mathcal{R}(0, -B)E \subseteq H^2(0,1) \subseteq L^\infty(0,1) = E_{\mathbb{1}},$$

by the Sobolev embedding in Theorem 5.3.7(b). Observing $A_0' = -A_0$, we similarly obtain $\mathcal{R}(1, A_0)'E' \subseteq (E')_{\mathbb{1}}$. Since $\mathcal{R}(1, A) = \mathcal{R}(0, -B)\mathcal{R}(1, A_0)$, it follows from Corollary 8.2.4 that $\pm\mathcal{R}(1, A) \preceq \mathbb{1} \otimes \mathbb{1}$. □

Example 8.3.5 (The Laplacian with non-local boundary conditions, revisited)**.** Consider the Laplace operator $\Delta_B : L^2(0,1) \supseteq \operatorname{dom}(\Delta_B) \to L^2(0,1)$ with non-local boundary conditions from Examples 5.4.3 and 6.3.4, whose domain is

$$\operatorname{dom}(\Delta_B) = \left\{ u \in H^2(0,1) : \begin{pmatrix} -u'(0) \\ u'(1) \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u(0) \\ u(1) \end{pmatrix} \right\}.$$

Then $\mathcal{R}(\cdot, \Delta_B)$ is uniformly eventually positive and negative with respect to $\mathbb{1} \otimes \mathbb{1}$ at $\mathrm{s}(\Delta_B)$.

Proof. We verify the assumptions of Theorem 8.3.2. Recall from Example 5.4.3 that $\mathrm{s}(\Delta_B) < 0$ and the explicit formula for the resolvent of Δ_B at 0 is given by

$$\mathcal{R}(0, \Delta_B)f(x) = \frac{1}{4}\int_0^1 f(z)\,\mathrm{d}z + \frac{1}{2}\int_x^1 \int_0^y f(z)\,\mathrm{d}z\,\mathrm{d}y + \frac{1}{2}\int_0^x \int_y^1 f(z)\,\mathrm{d}z\,\mathrm{d}y \tag{8.3.3}$$

for all $f \in L^2(0,1)$.

Step 1: If $0 \lneq f \in L^2(0,1)$, then formula (8.3.3) shows that

$$\mathcal{R}(0, \Delta_B)f(x) \geq \frac{1}{4}\int_0^1 f(z)\,\mathrm{d}z = \frac{1}{4}\big((\mathbb{1} \otimes \mathbb{1})f\big)(x)$$

for all $x \in [0,1]$, which implies $\mathcal{R}(0, \Delta_B) \succeq \mathbb{1} \otimes \mathbb{1}$. Applying Theorem 5.4.1 with $Q = \mathbb{1} \otimes \mathbb{1}$, we obtain that $\mathcal{R}(\mu, \Delta_B) \succeq \mathbb{1} \otimes \mathbb{1}$ for all $\mu \in (\mathrm{s}(\Delta_B), 0]$. This shows that $\mathcal{R}(\cdot, \Delta_B)$ is uniformly eventually positive with respect to $\mathbb{1}$ at $\mathrm{s}(\Delta_B)$.

Theorem 7.3.6 now implies that $\ker(\mathrm{s}(\Delta_B) - \Delta_B)$ is spanned by a vector $v \succeq \mathbb{1}$, and the dual eigenspace $\ker(\mathrm{s}(\Delta_B) - \Delta_B')$ contains a strictly positive functional φ. The property $v \succeq \mathbb{1}$ clearly shows that v is a quasi-interior point.

Step 2: Formula (8.3.3) also directly yields the estimate

$$\left|\mathcal{R}(0,\Delta_B)f\right| \le \left(\frac{1}{4}+\frac{1}{2}+\frac{1}{2}\right)\|f\|_{L^1(0,1)} = \frac{5}{4}\langle \mathbb{1}, |f|\rangle \mathbb{1}$$

for all $f \in L^2(0,1)$. Thus $\pm\mathcal{R}(0,\Delta_B) \preceq \mathbb{1}\otimes\mathbb{1}$.

We have thus verified all the conditions of Theorem 8.3.2 and conclude that $\mathcal{R}(\cdot,\Delta_B)$ is also uniformly eventually negative with respect to $\mathbb{1}\otimes\mathbb{1}$ at $\mathrm{s}(\Delta_B)$. □

8.4 Intermezzo: Hilbert space adjoints vs. Banach space duals

Our extensive use of form methods to construct linear operators on Hilbert spaces makes it worthwhile to spend a short intermezzo on clarifying the relation between dual operators on Banach space, adjoint operators on Hilbert spaces, and adjoints of sesquilinear forms. We use this in Example 8.5.2 in the next section. The definition of adjoint operators on a Hilbert space H is very similar to that of dual operators (Definition 3.1.5).

Definition 8.4.1 (Adjoint operators and forms)**.** Let V, H be complex Hilbert spaces.

(a) Let $A\colon H \supseteq \operatorname{dom}(A) \to H$ be densely defined linear operator. The **adjoint operator** $A^*\colon H \supseteq \operatorname{dom}(A^*) \to H$ is defined by

$$\begin{aligned}\operatorname{dom}\left(A^*\right) &:= \left\{x \in H \mid \exists\, y \in H\colon\ (x \mid Av) = \left(y \mid v\right)\ \forall\, v \in \operatorname{dom}(A)\right\}\\ A^*x &:= y,\end{aligned}$$

where y in the second line is the vector that occurs in the definition of $\operatorname{dom}(A^*)$.[3]

The operator A is called **self-adjoint** if $A^* = A$.

(b) Let $\mathfrak{a}\colon V\times V \to \mathbb{C}$ be a sesquilinear form. The form $\mathfrak{a}^*\colon V\times V\to\mathbb{C}$ given by $\mathfrak{a}^*(u,v) := \overline{\mathfrak{a}(v,u)}$ for all $u, v \in V$ is called the **adjoint form** of $\mathfrak{a}$.

Recall from Theorem 5.1.4(c) that a sesquilinear form $\mathfrak{a}$ is called **symmetric** if $\mathfrak{a}(u,v) = \overline{\mathfrak{a}(v,u)}$ for all $u, v \in V$. In other words, $\mathfrak{a}$ is symmetric if and only if $\mathfrak{a}^* = \mathfrak{a}$.

The adjoint and the dual of a Hilbert space operator are related as follows.

Proposition 8.4.2. *Let $A\colon H \supseteq \operatorname{dom}(A) \to H$ be densely defined linear operator on a complex Hilbert space H. Consider the mapping $J\colon H \to H'$, $x \mapsto (x \mid \cdot)$, which is an anti-linear isometric bijection by the Riesz–Fréchet theorem. Then $J\big(\operatorname{dom}(A^*)\big) = \operatorname{dom}\left(A'\right)$ and the following diagram commutes:*

$$\begin{array}{ccccc} H & \supseteq & \operatorname{dom}(A^*) & \xrightarrow{A^*} & H \\ \downarrow J & & \downarrow J & & \downarrow J \\ H' & \supseteq & \operatorname{dom}\left(A'\right) & \xrightarrow[A']{} & H' \end{array}$$

[3] Observe that y is unique by density of $\operatorname{dom}(A)$ in H.

Proof. This follows by chasing the definitions. □

Let (Ω, ν) be a σ-finite measure space. Observe that there are two common ways to identify $L^2(\Omega, \nu)$ with its dual space. When thinking mainly about Hilbert space, one typically identifies each $f \in L^2(\Omega, \nu)$ with the functional $\left(f \mid \cdot\right) = \int_\Omega \overline{f} \cdot \mathrm{d}\nu$. This identification is actually the anti-linear isomorphism from the Riesz-Fréchet representation theorem that we called J in Proposition 8.4.2. On the other hand, for $p \in [1, \infty)$ and $p' \in (1, \infty]$ with $1/p + 1/p' = 1$, it is more common to identify $L^{p'}(\Omega, \nu)$ with the dual space $\left(L^p(\Omega, \nu)\right)'$ by identifying each function $f \in L^{p'}(\Omega, \nu)$ with the functional $\int_\Omega f \cdot \mathrm{d}\nu$. Note that the isomorphism that maps f to this functional is linear rather than antilinear. However, for real-valued functions f both isomorphisms associate the same functional to f.

In particular, if A is a real operator and $\lambda \in \mathbb{R}$ then the real-valued elements of $\ker(\lambda - A^*)$ and $\ker(\lambda - A')$ coincide under those identifications. This is often useful to find out information on the dual operator A' since A^* can be described, again, by form method:

Proposition 8.4.3 (The adjoint operator via the adjoint form)**.** *Under the assumptions of Theorem 5.1.4, let $A\colon H \supseteq \operatorname{dom}(A) \to H$ be the operator associated to the form $\mathfrak{a}\colon V \times V \to \mathbb{C}$. Then A^* is the operator associated to the adjoint form $\mathfrak{a}^*$. In particular, if $\mathfrak{a}$ is symmetric, then A is self-adjoint.*

Giving in to the belief that this proposition will not appear too surprising, we refrain from discussing the proof and instead return to kernel estimates and eventual positivity.

8.5 Kernel estimates for resolvents via forms

We close this chapter with a tool to check the key assumption $\pm\mathcal{R}(\lambda_1, A) \preceq u \otimes \varphi$ in Theorem 8.3.2 if the operator A is associated to a sesquilinear form on L^2.

Proposition 8.5.1. *Let $H = L^2(\Omega, \nu)$ for a σ-finite measure space (Ω, ν) and let V be a complex Hilbert space such that V embeds continuously and densely into H. Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a bounded, real sesquilinear form on V that satisfies the ellipticity estimate*

$$\operatorname{Re} \mathfrak{a}(v, v) + \mu \|v\|_H^2 \geq \delta \|v\|_V^2$$

for some numbers $\mu \in \mathbb{R}$ and $\delta > 0$ and for all $v \in V$.

If $u \in E_+$ satisfies[4] *$V \subseteq H_u$, then the operator A associated with $\mathfrak{a}$ satisfies $\pm\mathcal{R}(\lambda, A) \preceq u \otimes u$ for one, hence all, $\lambda \in \rho(A) \cap \mathbb{R}$.*

Proof. By shifting the form we may assume that $\mu < 0$. Hence, $0 \in \rho(A)$ according to Theorem 5.1.4(b). Note that A is a real operator since $\mathfrak{a}$ is assumed to be real; hence, so is $\mathcal{R}(0, A)$. Let $x \in H$ and set $w := \mathcal{R}(0, A)x \in \operatorname{dom}(A) \subseteq V$. The inclusion map from V into H_u is continuous by the closed graph theorem, say with norm c. Hence,

$$\|w\|_{H_u}^2 \leq c^2 \|w\|_V^2 \leq \frac{c^2}{\delta} \operatorname{Re} \mathfrak{a}(w, w) \leq \frac{c^2}{\delta} |\mathfrak{a}(w, w)|$$

[4] This implies that u is a quasi-interior point since V is dense in H.

$$= \frac{c^2}{\delta} |(w \mid x)_H| \le \frac{c^2}{\delta} \|w\|_{H_u} \big(u \mid |x|\big)_H = \frac{c^2}{\delta} \|w\|_{H_u} \langle u, |x| \rangle,$$

so $\|w\|_{H_u} \le \frac{c^2}{\delta} \langle u, |x| \rangle$. In other words, $|\mathcal{R}(0,A)x| = |w| \le \frac{c^2}{\delta} \langle u, |x| \rangle \, u$. According to Theorem 8.2.1 this means that $\pm\mathcal{R}(0,A) \preceq u \otimes u$. □

Example 8.5.2 (The Neumann Laplacian on an interval)**.** Consider the sesquilinear form $\mathfrak{a}\colon H^1(0,\pi) \times H^1(0,\pi) \to \mathbb{C}$, $\mathfrak{a}(v,w) := \big(v' \mid w'\big)_{L^2}$ on $L^2(0,\pi)$. Its associated operator Δ_{Neu} acts as the weak second derivative on the domain

$$\operatorname{dom}(\Delta_{\text{Neu}}) = \big\{u \in H^2(0,\pi) : u'(0) = u'(\pi) = 0\big\}.$$

The operator Δ_{Neu} is called the **Neumann Laplace operator** and has the following properties.

(a) Δ_{Neu} has compact resolvent.

(b) $\operatorname{s}(\Delta_{\text{Neu}}) = 0$ is an eigenvalue and $\ker \Delta_{\text{Neu}}$ and $\ker \Delta'_{\text{Neu}}$ are both spanned by $\mathbb{1}$.

(c) For every $\lambda > 0$ one has $\mathcal{R}(\lambda, \Delta_{\text{Neu}}) \ge 0$.

(d) The resolvent $\mathcal{R}(\cdot, \Delta_{\text{Neu}})$ is uniformly eventually positive and negative with respect to $\mathbb{1} \otimes \mathbb{1}$ at 0.

Proof. The fact that Δ_{Neu} has the claimed domain and acts as the weak second derivative, is a special case of Exercise 5.6(a). Let us show that Δ_{Neu} has the claimed properties.

(b) For each $\mu > 0$, we have

$$\operatorname{Re} \mathfrak{a}(v,v) + \mu \|v\|_{L^2}^2 = \|v'\|_{L^2}^2 + \mu \|v\|_{L^2}^2 \ge \min\{1,\mu\} \|v\|_{H^1}^2$$

for all $v \in H^1(0,\pi)$. Therefore by Theorem 5.1.4, the associated operator is closed and densely defined, and $\operatorname{s}(\Delta_{\text{Neu}}) \le 0$. In fact, $\operatorname{s}(\Delta_{\text{Neu}}) = 0$, as one can easily check that 0 is an eigenvalue and the corresponding eigenspace is spanned by $\mathbb{1}$. Since the form $\mathfrak{a}$ is symmetric, the operator Δ_{Neu} is self-adjoint (Proposition 8.4.3) and hence, the dual eigenspace is also spanned by $\mathbb{1}$.

(a) The embedding $H^1(0,\pi) \hookrightarrow L^2(0,\pi)$ is compact by Theorem 6.3.1 and we have shown in the proof of (b) that the assumptions of Theorem 5.1.4 are fulfilled. It follows that Δ_{Neu} has compact resolvent by Proposition 6.2.10(b).

(c) Recall from Example 4.1.4(d) that $H^1(0,\pi;\mathbb{R}) = H^1(0,\pi) \cap L^2((0,\pi);\mathbb{R})$ is a sublattice of $L^2((0,\pi);\mathbb{R})$ and

$$\mathfrak{a}(v^-, v^+) = \big(-\mathbb{1}_{v<0}\, v' \mid \mathbb{1}_{v>0}\, v'\big)_{L^2} = 0$$

for all $v \in H^1(0,\pi;\mathbb{R})$. The Beurling–Deny criterion (Theorem 5.1.7) therefore implies the assertion.

(d) By the one-dimensional Sobolev embeddings in Theorem 5.3.7(b), we know that $H^1(0,\pi) \subseteq L^\infty(0,\pi) = L^2(0,\pi)_{\mathbb{1}}$. As a result $\pm\mathcal{R}(\lambda, \Delta_{\text{Neu}}) \preceq \mathbb{1} \otimes \mathbb{1}$ for all $\lambda > 0$ by Proposition 8.5.1. Thus, all assumptions of Theorem 8.3.2 are fulfilled with $u = \varphi = \mathbb{1}$, whence the assertion follows. □

Exercises for Chapter 8

Exercise 8.1 (AL-spaces generated by functionals vs. principal ideals)**.** Let E be a Banach lattice and $\varphi \in E'$ a strictly positive functional. Show that the Banach lattices $(E^{\varphi})'$ and $(E')_{\varphi}$ are isomorphic.

More precisely, show that $J\colon (E^{\varphi})' \to E'$, $\psi \mapsto \psi|_E$ maps the dual space $(E^{\varphi})'$ bijectively to the principal ideal $(E')_{\varphi}$ and that $J\psi \geq 0$ if and only if $\psi \geq 0$.

Exercise 8.2 (A fourth order operator on an interval, continued)**.** Consider the fourth order differential operator $A\colon L^2(0,1) \supseteq \operatorname{dom}(A) \to L^2(0,1)$ from Exercise 7.2. We know from that exercise that $\mathcal{R}(\cdot, A)$ is individually eventually positive with respect to $\mathbb{1}$ at the spectral bound 0. Now we improve this result.

(a) Show that $\mathcal{R}(\cdot, A)$ is uniformly eventually positive with respect to $\mathbb{1} \otimes \mathbb{1}$ at 0.

(b) Is $\mathcal{R}(\cdot, A)$ also uniformly eventually negative with respect to $\mathbb{1} \otimes \mathbb{1}$ at 0?

Exercise 8.3 (Yet another fourth order operator on an interval)**.** Consider the Dirichlet Laplace operator $\Delta_{\mathrm{Dir}}\colon L^2(0,\pi) \supseteq \operatorname{dom}(\Delta_{\mathrm{Dir}}) \to L^2(0,\pi)$. Define $A := -\Delta_{\mathrm{Dir}}^2$.

(a) Show that $\sigma(A) \subseteq (-\infty, 0)$ and that A has compact resolvent.

(b) Prove that $\mathcal{R}(\cdot, A)$ is uniformly eventually positive and negative with respect to $\sin \otimes \sin$ at $\mathrm{s}(A)$.

Exercise 8.4 (A first order operator on a space of continuous functions)**.** In this exercise we revisit the differential operator A_0 from Example 8.3.3, but now on the space $\mathrm{C}([0,1])$. Let $A_0\colon \mathrm{C}([0,1]) \supseteq \operatorname{dom}(A_0) \to \mathrm{C}([0,1])$ be given by

$$\begin{aligned} \operatorname{dom}(A_0) &:= \left\{ f \in \mathrm{C}^1([0,1]) : f(0) = f(1) \right\} \\ A_0 f &:= f'. \end{aligned}$$

(a) Show that A_0 has compact resolvent, that $\sigma(A_0) = 2\pi \mathrm{i}\mathbb{Z}$, and that the resolvent at $\mu \in \rho(A_0)$ is given by the integration against the kernel from formula (8.3.1).

(b) Find a strictly positive functional $\varphi \in \mathrm{C}([0,1])$ that spans the kernel $\ker A'$ of the dual operator. Is φ is a quasi-interior point of $\mathrm{C}([0,1])'_+$?

(c) Prove that $\mathcal{R}(\mu, A_0) \preceq -\mathbb{1} \otimes \varphi$ for all $\mu \in (-\infty, 0)$ and $\mathcal{R}(\mu, A_0) \succeq \mathbb{1} \otimes \varphi$ for all $\mu \in (0, \infty)$ where φ is the functional from (b).

Notes for Chapter 8

Sufficient conditions for uniform eventual positivity

Theorem 8.3.2 is a special case of the main result of [AG22], which was in turn inspired by earlier results of Takáč [Tak96]. In fact, one can show the same conclusion as in Theorem 8.3.2 if the assumption $\pm\mathcal{R}(\lambda_1, A) \preceq u \otimes \varphi$ is replaced with the one-sided estimate $\mathcal{R}(\lambda_1, A) \succeq -u \otimes \varphi$ along with the weaker assumptions $\operatorname{dom}(A^m) \subseteq E_u$ and $\operatorname{dom}\big((A')^m\big) \subseteq (E')_\varphi$ for some $m \in \mathbb{N}$; see [AG22, Theorem 1.2].

For concrete differential operators, the domain embeddings usually follow from elliptic regularity results (up to the boundary) combined with Sobolev embedding theorems – a theme that we have already used multiple times for the Dirichlet Laplace operators. The lower estimate $\mathcal{R}(\lambda_1, A) \succeq -u \otimes \varphi$ can sometimes be shown for higher order differential operators as a consequence of the observation that, while the integral kernel of the resolvent – i.e. the **Green's function** – of such operators is not positive, its singularity typically is. Such concrete kernel estimates are, for instance, shown in [DMS05, Theorem 1.5], [GR10, Theorem 1], and [Pul15, Theorem 4.1]. It would be very desirable to have an general operator theoretic explanations for this kind of behaviour, but we are currently not aware of any such abstract explanation.

Automatic compactness

The assumption in Theorem 8.3.2 that λ be a pole of the resolvent, is actually redundant. In fact, the assumption $\pm\mathcal{R}(\lambda_1, A) \preceq u \otimes \varphi$ implies that $\mathcal{R}(\lambda_1, A)^3$ is compact. Indeed, it is a classical result in Banach lattice theory that if $0 \le S_k \le T_k$ for linear operators and for indices $k \in \{1,2,3\}$, then compactness of all T_k imply that $S_3 S_2 S_1$ is compact [MN91, Corollary 3.7.14]. From this one can readily derives that if $\pm S \preceq T$ and T is compact, then S^3 is compact; see [AG22, Corollary 2.7] for details. Since $\mathcal{R}(\lambda_1, A)^3$ is compact, one can derive from analytic Fredholm theory that every spectral value of A is a pole of $\mathcal{R}(\cdot, A)$.

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