Chapter 7

Criteria for eventual positivity of resolvents: the individual case

In Chapter 6, we introduced some flexibility in the definition of eventually positive resolvents (Definition 6.1.1), enabling us to consider both the weak inequality ≥ 0 as well as a lower bound $\succeq u$ with respect to some non-zero positive vector u. We refer to the latter case informally as **strong positivity**, in analogy with the property $\succeq \mathbb{1}$ in finite dimensions. The present and the subsequent chapter are devoted to a study of eventual strong positivity of resolvents.

7.1 Banach lattice overture: Principal ideals and quasi-interior points

It turns out that a useful generalisation of the constant vector $\mathbb{1} \in \mathbb{R}^n$ is the notion of a **quasi-interior point** in a Banach lattice.

Definition 7.1.1 (Principal ideals and quasi-interior points)**.** Let E be a Banach lattice and let $u \in E_+$.

(a) The set $E_u := \{x \in E : |x| \leq u\}$ is called the **principal ideal** in E generated by u.[1] The mapping $\|\cdot\|_{E_u} : E_u \to [0,\infty)$ given by

$$\|x\|_{E_u} := \inf\{c \in [0,\infty) : |x| \leq cu\}$$

is called the **gauge norm** with respect to u.

(b) The vector u is called a **quasi-interior point** of E_+ if E_u is dense in E.

If the surrounding Banach lattice and its positive cone are clear from context, we will sometimes say "u is a quasi-interior point" as a shorthand for "u is a quasi-interior point of E_+."

[1] Of course, principal ideals are ideals.

The following example is simple but very instructive.

Example 7.1.2. Let (Ω, ν) be a finite measure space, so that $\mathbb{1}_\Omega \in L^p(\Omega,\nu)$ for all $p \in [1,\infty]$. Then it is easy to see that $(L^p(\Omega,\nu))_{\mathbb{1}_\Omega} = L^\infty(\Omega,\nu)$ for all $p \in [1,\infty]$, and the corresponding gauge norm is precisely the essential supremum norm.

Note that if $p \neq \infty$, then $L^\infty(\Omega,\nu)$ is not a closed ideal in $L^p(\Omega,\nu)$.

Below, we use the not particularly surprising observation that an element x of a Banach lattice is 0 if $|x| = 0$. In the real case this follows from $0 \le x^+, x^- \le |x|$ and $x = x^+ - x^-$. In the complex case it then follows from Proposition 4.2.7.

Proposition 7.1.3. *Let E, F be Banach lattices over the same scalar field, let $u \in E_+$ be a quasi-interior point, and let $0 \le T \in \mathcal{L}(E,F)$. If $Tu = 0$, then $T = 0$.*

Proof. The positivity of T implies that $|Tx| \le T|x| \le Tu = 0$ (Proposition 4.3.2) and thus $Tx = 0$ for all $x \in E$ that satisfy $|x| \le u$. But the span of such x is dense in E owing to the fact that u is a quasi-interior point. □

Proposition 7.1.4. *Let E be a Banach lattice. A vector $u \in E_+$ is a quasi-interior point if and only if $\langle \psi, u \rangle > 0$ for each $0 \lneq \psi \in E'$.*

Proof. "⇒": This follows from Proposition 7.1.3.

"⇐": If u is not a quasi-interior point of E_+, then by the Hahn-Banach theorem, there exists $\psi \in E' \setminus \{0\}$ such that $\langle \psi, x \rangle = 0$ for each $x \in \overline{E_u}$. Thus, $0 \lneq |\psi| \in E'$ satisfies

$$\langle |\psi|, u \rangle = \sup_{|x| \le u} |\langle \psi, x \rangle| = 0$$

due to the Riesz-Kantorovich formula (Theorem 4.4.2). □

Examples 7.1.5. The following examples are discussed in detail in Exercise 7.1.

(a) Let (Ω, μ) be a σ-finite measure space. The quasi-interior points of $L^p(\Omega,\mu)_+$ for $p \in [1,\infty)$ are exactly those functions in $L^p(\Omega,\mu)$ that are strictly positive almost everywhere.

(b) Let (Ω, μ) be a σ-finite measure space. The quasi-interior points of $L^\infty(\Omega,\mu)_+$ are exactly those $f \in L^\infty(\Omega,\mu)$ that satisfy $f \succeq \mathbb{1}_\Omega$.

(c) Let K be a compact metric space[2]. A function $u \in C(K)_+$ is a quasi-interior point if and only if $u(x) > 0$ for all $x \in K$ if and only if $u \succeq \mathbb{1}_K$.

(d) Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open[3]. The quasi-interior points of $C_0(\Omega)_+$ are precisely those functions $u \in C_0(\Omega)$ that satisfy $u(x) > 0$ for all $x \in \Omega$.

[2] Or more generally, a compact Hausdorff topological space.

[3] Or more generally, let $\emptyset \neq \Omega$ be a locally compact Hausdorff topological space.

(e) In particular, in finite dimensions the quasi-interior points are exactly the strongly positive vectors; see Definition 1.2.3.

Proposition 7.1.6. *Let E be a Banach lattice and $u \in E_+$. The principal ideal E_u equipped with the gauge norm is a Banach lattice that embeds continuously into E.*

For the proof, one can first show the proposition in the real case and then derive the complex case from it. Writing down all the details is a bit tedious, but does not require any surprising ideas, so we refrain from doing so.

7.2 Strong positivity properties of the Dirichlet Laplacian

In this section, we revisit the maximum principle of Chapter 5, and demonstrate how it may be used to prove strong positivity properties for certain PDEs. For simplicity, we once again focus on the Dirichlet Laplacian as the leading example. While this material is part of classical PDE theory, it falls neatly into our abstract framework nonetheless.

Theorem 7.2.1 (Location of strict maxima)**.** *Let (M, d) be a metric space and let $\emptyset \neq S \subseteq M$ be relatively compact. Let $D \subseteq \mathrm{C}(\overline{S}; \mathbb{R})$ be a vector subspace such that $\mathbb{1} := \mathbb{1}_{\overline{S}} \in D$ and let $A\colon D \to \mathbb{R}^S$ be a linear map with the same properties as in Theorem 5.2.1, i.e.*

(1) *The map A satisfies the positive minimum principle on S, i.e. for each $x \in S$ and each function $0 \le u \in D$ one has the implication*

$$u(x) = 0 \qquad \Longrightarrow \qquad (Au)(x) \ge 0.$$

(2) *One has $A\mathbb{1} \le 0$ and there exists a function $0 \le w \in D$ with $(Aw)(x) > 0$ for all $x \in S$.*

(3) *Let $x_0 \in \partial S$ and assume that the function w from assumption (2) vanishes at all points in ∂S that are sufficiently close to x_0*

Let $v \in D$ attain at least one value in $[0, \infty)$ and satisfy $Av \ge 0$ in S. If v has a strict global maximum at x_0, then x_0 is not in S.

Proof. Assume to the contrary that $x_0 \in S$. Since, by assumption, v has a strict global maximum at x_0 and w vanishes at all points of ∂S that are close to x_0 (assumption (3)), we can find a number $\varepsilon > 0$ such that $v(x_0) \ge v(x) + \varepsilon w(x)$ for all $x \in \partial S$. Set $h := v + \varepsilon w - v(x_0)\,\mathbb{1}$. Then $h(x) \le 0$ for all $x \in \partial S$ and $h(x_0) = 0$. Moreover,

$$(Ah)(x) = (Av)(x) + \varepsilon (Aw)(x) - v(x_0)(A\mathbb{1})(x) \ge (Av)(x) + \varepsilon(Aw)(x) > 0$$

for all $x \in S$, where we used for the first inequality that $v(x_0) \ge 0$ and $(A\mathbb{1})(x) \le 0$. It follows from Theorem 5.2.1, applied to the function h, that h attains its maximum on ∂S. Since $h \le 0$ on ∂S, we conclude that $h \le 0$ in S.

Hence, the function $0 \le -h \in D$ satisfies $-h(x_0) = 0$ but $\big(A(-h)\big)(x_0) = -(Ah)(x_0) < 0$, contradicting the positive minimum principle (assumption (1)) at the point $x_0 \in S$. □

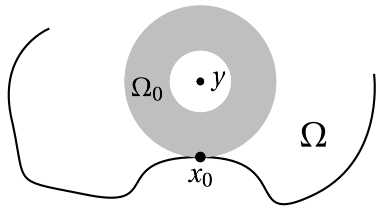

Figure 7.2.1: Geometric conditions in the Hopf lemma.

As an application of the previous theorem, we prove a simple version of the classical Hopf boundary lemma. It is perhaps surprising that we obtain this result in the same framework as for the classical maximum principle, and thus we state it as an example.

Example 7.2.2 (Hopf boundary lemma)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open[4] and let $c \leq 0$ be a real number. Let $v \in \mathrm{C}^1(\overline{\Omega};\mathbb{R}) \cap \mathrm{C}^2(\Omega;\mathbb{R})$ be such that $\Delta v(x) + cv(x) \geq 0$ for all $x \in \Omega$ and assume that $x_0 \in \partial\Omega$ has the following properties:

- **Strict maximum at x_0:** One has $v(x_0) \geq 0$ and $v(x_0) > v(x)$ for all $x \in \Omega$.
- **Interior ball condition:** There exists an open ball $B \subseteq \Omega$ with $x_0 \in \partial B$.

If ν denotes the outer unit normal of the ball B, then $\partial_\nu v(x_0) > 0$.

Proof. Geometric setup: By the interior ball condition, there exist $y \in \Omega$ and $R > 0$ such that $B_{<R}(y) \subset \Omega$ and $x_0 \in \partial B_{<R}(y)$. By decreasing R and moving y a bit towards x_0 if necessary, we may assume that $\partial\Omega \cap \partial B_{<R}(y) = \{x_0\}$. Choose an arbitrary $r \in (0, R)$ and consider the annular region $\Omega_0 = B_{<R}(y) \setminus B_{\leq r}(y)$; see Figure 7.2.1 for a visual aid. Set $S := \Omega_0 \cup \{x_0\}$.

Let D be the space of restrictions of all functions in $\mathrm{C}^1(\overline{\Omega};\mathbb{R}) \cap \mathrm{C}^2(\Omega;\mathbb{R})$ to $\overline{S}$. Clearly $\mathbb{1}_{\overline{S}} \in D$. Define a linear operator $A\colon D \to \mathbb{R}^S$ by

$$(Af)(x) := \begin{cases} \Delta f(x) + cf(x) & x \in \Omega_0 \\ -\partial_\nu f(x) & x = x_0 \end{cases} \qquad \forall f \in D.$$

We verify that the assumptions of Theorem 7.2.1 are satisfied:

(1) Suppose $0 \leq f \in D$ such that $f(x) = 0$ for some $x \in S$. We want to show $(Af)(x) \geq 0$ If $x = x_0$, then $(Af)(x) = -(\partial_\nu f)(x) \geq 0$ (observe that $-\partial_\nu$ is the inward normal derivative). On the other hand, if $x \in \Omega_0$, the conclusion $(Af)(x) \geq 0$ follows from the fact that the operator $\Delta + c$ satisfies the positive minimum principle in Ω_0 (verified in the proof of Example 5.2.2).

[4] Not necessarily bounded.

(2) Of course, $A\mathbb{1}_{\overline{S}} \le 0$. For all $x \in \overline{S}$, define $w\colon \overline{S} \to \mathbb{R}$ by

$$w(x) := e^{-\alpha\|x-y\|_2^2} - e^{-\alpha R^2}$$

for some $\alpha > 0$ to be determined later. We have $0 \le w \in D$, and since $c \le 0$, we deduce

$$Aw(x) = \left(4\alpha^2 \|x-y\|_2^2 - 2\alpha n + c\right)e^{-\alpha\|x-y\|_2^2} - ce^{-\alpha R^2} \ge \left(4\alpha^2 r^2 - 2\alpha n + c\right)e^{-\alpha\|x-y\|_2^2}$$

for all $x \in \Omega_0$. Hence we can choose $\alpha > 0$ sufficiently large so that $(Aw)(x) > 0$ for all $x \in \Omega_0$. We further note that $\nu(x_0) = \frac{x_0 - y}{R}$, which yields

$$(Aw)(x_0) = -\partial_\nu w(x_0) = -\nabla w(x_0) \cdot \nu(x_0) = 2\alpha R e^{-\alpha R^2} > 0.$$

Therefore $(Aw)(x) > 0$ for all $x \in S$.

(3) Note that w vanishes on $\partial B_{<R}(y)$, in particular, it vanishes at all points in ∂S sufficiently close to x_0.

Finally, the restriction of v to $\overline{S}$ lies in D, attains a positive value and a strict global maximum at $x_0 \in S$. Thus, the location of the strict maximum in Theorem 7.2.1 implies that $Av \not\ge 0$ on S. But, $Av \ge 0$ on Ω_0 by assumption. Consequently, $\partial_\nu v(x_0) = -(Av)(x_0) > 0$. □

Corollary 7.2.3 (Strong maximum principle for the Laplace operator)**.** *Let $\emptyset \ne \Omega \subseteq \mathbb{R}^n$ be connected and open*[5] *and let $c \in (-\infty, 0]$. Let $v \in \mathrm{C}^2(\Omega;\mathbb{R})$ satisfy $\Delta v(x) + cv(x) \ge 0$ for all $x \in \Omega$. If v attains a positive maximum at an interior point of Ω, then v is constant in Ω.*

Proof. Assume for contradiction that v is non-constant and attains a positive maximum $M \ge 0$ at an interior point of Ω. The set $\Omega^- := \{x \in \Omega : v(x) < M\}$ is non-empty and open, and because Ω is connected, so $\partial\Omega^- \cap \Omega \ne \emptyset$. Choose a point $y \in \Omega^-$ such that $\operatorname{dist}(y, \partial\Omega^-) < \operatorname{dist}(y, \partial\Omega)$, and let B be the largest ball centred at y contained entirely in Ω^-. By choice of B, the boundary ∂B touches $\partial\Omega^-$, so $v(x_0) = M$ for some point $x_0 \in \partial B$, while $v(x) < M$ for all $x \in B$. In other words, v has a strict maximum at x_0.

As B trivially satisfies the internal ball condition at x_0, the Hopf boundary lemma (Example 7.2.2) ensures that $\partial_\nu v(x_0) > 0$. But x_0 is an interior point of Ω and v attains a maximum there, so that $\nabla v(x_0) = 0$, a contradiction. □

Example 7.2.4 (Strong positivity for the Dirichlet Laplacian)**.** Let $\emptyset \ne \Omega \subseteq \mathbb{R}^n$ be open, bounded, and connected. We assume that Ω has C^{2k} boundary for some integer $k > \frac{n}{4} + 1$. Consider the quasi-interior point u of $L^2(\Omega)_+$ given by $u(x) := \operatorname{dist}(x, \partial\Omega)$ for all $x \in \Omega$.

The resolvent of the Dirichlet Laplace operator $\Delta_{\mathrm{Dir}}\colon L^2(\Omega) \supseteq \operatorname{dom}(\Delta_{\mathrm{Dir}}) \to L^2(\Omega)$ has the following properties at every $\lambda > \mathrm{s}(\Delta_{\mathrm{Dir}})$:

(a) For every $f \in L^2(\Omega)$, the function $\mathcal{R}(\lambda, \Delta_{\mathrm{Dir}})^k f$ is in $\mathrm{C}^2(\overline{\Omega})$ and thus continuous on $\overline{\Omega}$ and satisfies $\left|\mathcal{R}(\lambda, \Delta_{\mathrm{Dir}})^k f\right| \le u$. In particular $\mathcal{R}(\lambda, \Delta_{\mathrm{Dir}})^k$ maps into the principal ideal $L^2(\Omega)_u$.

[5] Not necessarily bounded.

(b) In addition, if $0 \lneq f \in L^2(\Omega)$, then $u \preceq \mathcal{R}(\lambda, \Delta_{\mathrm{Dir}})f$.

Proof. (a) Let $f \in L^2(\Omega)$. We first observe that $g := \mathcal{R}(\lambda, \Delta_{\mathrm{Dir}})^k f \in \mathrm{C}^2(\overline{\Omega})$. Indeed, one has $\mathcal{R}(\lambda, \Delta_{\mathrm{Dir}})^k L^2(\Omega) \subseteq H^{2k}(\Omega) \subseteq \mathrm{C}^2(\overline{\Omega})$, where the first inclusion follows from elliptic regularity for the Dirichlet Laplacian (Theorem 5.3.2), and the second from the Sobolev embedding theorem 5.3.4 since Ω has C^{2k} boundary and $2k > \frac{n}{2} + 2$.

In particular, g is continuous on $\overline{\Omega}$, so it remains to show that $|g| \preceq u$. To this end, fix a point $x \in \Omega$.

The compactness of $\partial\Omega$ yields the existence of some $x_0 \in \partial\Omega$ such that $\operatorname{dist}(x, \partial\Omega) = \|x - x_0\|_2$. Observe that $g(x_0) = 0$; indeed, g vanishes on all of $\partial\Omega$ since it is a continuous element of $\operatorname{dom}(A)$ and thus of $H_0^1(\Omega)$ (cf. Proposition 5.3.5(b)). Since $g \in \mathrm{C}^1(\overline{\Omega})$, the fundamental theorem of calculus yields

$$g(x) = \int_0^1 (\nabla g)\big((1-t)x_0 + tx\big)\,\mathrm{d}t \cdot (x - x_0),$$

which implies the estimate

$$\big|g(x)\big| \le \big\|\nabla g\big\|_{\mathrm{C}(\overline{\Omega})} \|x - x_0\|_2 = \big\|\nabla g\big\|_{\mathrm{C}(\overline{\Omega})} \operatorname{dist}(x, \partial\Omega).$$

Hence $|g| \preceq u$ as claimed.

(b) Now assume that $0 \lneq f \in L^2(\Omega)$. We show the lower bound $\mathcal{R}(\lambda, \Delta_{\mathrm{Dir}})f \succeq u$ in three steps.

Step 1: We assume in addition that $\lambda \ge 0$ and show that $g := \mathcal{R}(\lambda, \Delta_{\mathrm{Dir}})^k f \succeq u$.

We already know that $\mathcal{R}(\lambda, \Delta_{\mathrm{Dir}}) \ge 0$ (either from Example 5.3.6 or by the Beurling-Deny criterion of Theorem 5.1.7), so $g \ge 0$. Next, we show that $g(x) > 0$ for every $x \in \Omega$, so fix such an x and assume that $g(x) = 0$.

One has $(\Delta_{\mathrm{Dir}} - \lambda)(-g) = \mathcal{R}(\lambda, \Delta_{\mathrm{Dir}})^{k-1} f \ge 0$. Since $\lambda \ge 0$ and since $g \in \mathrm{C}^2(\overline{\Omega})$ according to the proof of (a), the strong maximum principle (Corollary 7.2.3) is applicable to $-g$. As $-g \le 0$ and $-g(x) = 0$, it follows that $-g$ is constant and thus $g = 0$. But this is absurd since $\mathcal{R}(\lambda, \Delta_{\mathrm{Dir}})^k$ is injective and $f \ne 0$.

Now we consider the behaviour of g close to $\partial\Omega$ separately from the behaviour away from the boundary. To this end, let $\delta > 0$ be a number that we determine later and consider the compact set

$$\Omega_\delta := \{x \in \Omega : u(x) \ge \delta\}.$$

One has $g|_{\Omega_\delta} \succeq \mathbb{1}_{\Omega_\delta} \succeq u|_{\Omega_\delta}$ since $g(x) > 0$ for all $x \in \Omega$, so it remains to show that $g|_{\Omega \setminus \Omega_\delta} \succeq u|_{\Omega \setminus \Omega_\delta}$.

To this end we use the following geometric fact: since Ω has C^m boundary for some $m \ge 2$, the interior ball condition is satisfied at every boundary point $x_0 \in \partial\Omega$[6]. By the Hopf boundary lemma (Example 7.2.2), we have $-\partial_\nu g(x_0) > 0$ for all

[6]An intuitive 'proof' can be deduced from the second-order Taylor expansion of a C^m function; see [GT01, pp. 354–355] for a proof using the notion of boundary curvatures.

$x_0 \in \partial\Omega$. Hence, by the compactness of $\partial\Omega$ and the continuity of ∇g on $\overline{\Omega}$, there exists a constant $c > 0$ such that $-\partial_\nu g \geq c$ on $\partial\Omega$.

Now let $x \in \Omega \setminus \Omega_\delta$. Choose $x_0 \in \partial\Omega$ such that $u(x) = \|x - x_0\|_2 < \delta$. Since $g(x_0) = 0$, Taylor's formula with remainder gives the estimate

$$g(x) = \underbrace{\nabla g(x_0)^{\mathrm{T}}(x - x_0)}_{= -\partial_\nu g(x_0)\|x - x_0\|_2} + \int_0^1 (1-t)(x-x_0)^{\mathrm{T}}(Hg)\big((1-t)x_0 + tx\big)(x - x_0)\,\mathrm{d}t$$
$$\geq c\,\|x - x_0\|_2 - \frac{1}{2}\,\|Hg\|_{\mathrm{C}(\overline{\Omega})}\,\|x - x_0\|_2^2 \geq \Big(c - \frac{\delta}{2}\,\|Hg\|_{\mathrm{C}(\overline{\Omega})}\Big)u(x),$$

where Hg is the Hessian matrix of g. So if c is chosen sufficiently small, then $g(x) \geq \frac{c}{2}u(x)$ holds for all $x \in \Omega \setminus \Omega_\delta$.

Step 2: Consider a number $\mu \in (\mathrm{s}(\Delta_{\mathrm{Dir}}), \lambda)$. If μ is sufficiently close to λ, the Taylor series expansion of the resolvent (Proposition 3.3.2(a)) gives

$$\mathcal{R}(\mu, \Delta_{\mathrm{Dir}})f = \sum_{j=0}^{\infty} (\lambda - \mu)^j \mathcal{R}(\mu, \Delta_{\mathrm{Dir}})^{j+1} f \geq (\lambda - \mu)^{k-1} g \geq u,$$

where we used that $\mathcal{R}(\mu, \Delta_{\mathrm{Dir}}) \geq 0$. Finally, one can repeat the argument from the proof of Theorem 5.4.1 to see the estimate $\mathcal{R}(\mu, \Delta_{\mathrm{Dir}})f \succeq u$. □

We will use Example 7.2.4 at the end of the next section (Example 7.3.8) to obtain further knowledge about the eigenspace associated to the spectral bound of Δ_{Dir}.

7.3 Characterisation of individual eventual strong positivity

After the deep dive into the positivity properties of the Dirichlet Laplacian in the previous section, we continue with developing the theory of eventually positive resolvents – and we shall meet Δ_{Dir} again before the end of the section. From Theorem 6.3.3 we know that, for semisimple eigenvalues, eventual positivity guarantees that the spectral projection is positive. For eventual positivity with respect to a quasi-interior point, it turns out even a characterization can be given in terms of the spectral projection (Theorem 7.3.6).

Notation 7.3.1. Let X be a Banach space. For $u \in X$ and $\varphi \in X'$, we define the operator

$$u \otimes \varphi := \langle \varphi, \cdot \rangle\, u \in \mathcal{L}(X).$$

Clearly, the operator $u \otimes \varphi$ has rank 1 unless $u = 0$ or $\varphi = 0$. It is not difficult to check that its operator norm is $\|u \otimes \varphi\| = \|u\|\,\|\varphi\|$ and that the operator is a projection if and only if $\langle \varphi, u \rangle = 1$.

A functional $\varphi \in E'$ on a Banach lattice E is called **strictly positive** if $\langle \varphi, f \rangle > 0$ for all $0 \lneq f \in E$.

Proposition 7.3.2. *Let E be a Banach lattice, let $P \in \mathcal{L}(E)$ be a projection, let $0 \lneq u \in E$ and assume that $Pf \succeq u$ for all $0 \lneq f \in E$. Then there is a strictly positive functional $\varphi \in E'$ such that $P = (Pu) \otimes \varphi$.*

Proof. First observe that the assumption implies $P \gneq 0$ and that $Pu \succeq u$. Moreover, for each $0 \lneq f \in E$ one has $Pf \succeq u$ by assumption, and hence, by applying the projection P again, $Pf \succeq Pu$.

Now consider a vector $0 \le w \in \operatorname{rg} P$. We show that w is a multiple of Pu. Note that $w - tPu \not\geq 0$ for all sufficiently large $t > 0$. Indeed, if we could find numbers $t_n \to \infty$ such that $w - t_n Pu \ge 0$ for all n, then dividing by t_n and letting $n \to \infty$ would yield $0 \ge Pu$, which is absurd as Pu is positive and non-zero. On the other hand, $w - tPu \ge 0$ for $t \le 0$. Hence, there exists a maximal $t_0 \in \mathbb{R}$ for which $w - t_0 Pu \ge 0$. If $w - t_0 Pu \gneq 0$, it follows that $w - t_0 Pu \succeq Pu$, which contradicts the maximality of t_0. So $w - t_0 Pu = 0$.

Since E_+ spans E, the space $\operatorname{rg} P$ is spanned by its positive elements, so it is actually spanned by Pu. As P is a projection, it follows that there is a non-zero functional $\varphi \in E'$ such that $P = (Pu) \otimes \varphi$. Since P and Pu are positive, so is φ. Finally, as $Px \gneq 0$ for every $x \gneq 0$, it follows that φ is strictly positive. □

We now look at a couple of auxiliary results that aid in the proof of Theorem 7.3.6.

Lemma 7.3.3. *Let A be a closed operator on a complex Banach space X and let $\lambda \in \sigma(A)$.*

(a) *If λ is a geometrically simple eigenvalue and there exist $v \in \ker(\lambda - A)$ and $\psi \in \ker(\lambda - A')$ such that $\langle \psi, v \rangle \neq 0$, then λ is algebraically simple.*

(b) *Suppose that $X = E$ is a Banach lattice and $\lambda \in \mathbb{R}$ is a pole of the resolvent $\mathcal{R}(\cdot, A)$ of order $p \in \mathbb{N}$ such that the coefficient Q_{-p+1} of $(\mu - \lambda)^p$ in the Laurent series expansion of $\mathcal{R}(\cdot, A)$ is positive.*

   If $\ker(\lambda - A)$ contains a quasi-interior point of E_+, then $p = 1$.

Proof. (a) The proof of Lemma 1.2.8 carries over mutatis mutandis.

(b) Let $v \in \ker(\lambda - A)$ be a quasi-interior point of E_+. Since Q_{-p+1} is positive (by assumption) and non-zero (by Theorem 6.2.6(a)), Proposition 7.1.3 implies that $Q_{-p+1} v \neq 0$. Employing $v \in \ker(\lambda - A)$, we obtain that

$$\lim_{\mu \to \lambda} (\mu - \lambda)^{p-1} v = \lim_{\mu \to \lambda} (\mu - \lambda)^p \mathcal{R}(\mu, A) v = Q_{-p+1} v \neq 0.$$

As a consequence, $p = 1$. □

Lemma 7.3.4. *Let $\lambda \in \mathbb{R}$ be a spectral value of a closed operator $A\colon X \supseteq \operatorname{dom}(A) \to X$ on a complex Banach space X. If λ is a first order pole of the resolvent $\mathcal{R}(\cdot, A)$, then*

$$\lim_{\mu \to \lambda} \left\| (\mu - \lambda)^m \mathcal{R}(\mu, A)^m - P \right\|_{\mathcal{L}\left(X, \operatorname{dom}(A^m)\right)} = 0 \quad \textit{for all } m \in \mathbb{N};$$

where P denotes the spectral projection of A associated to λ.

Proof. As λ is a first order pole, $\lim_{\mu \to \lambda} (\mu - \lambda)^m \mathcal{R}(\mu, A)^m = P$ in $\mathcal{L}(X)$ and $\operatorname{rg} P = \ker(\lambda - A)$ by Theorem 6.2.6(b) and (c). Therefore using $A\mathcal{R}(\mu, A) = \mu \mathcal{R}(\mu, A) - \mathrm{id}$, we obtain

$$A^m (\mu - \lambda)^m \mathcal{R}(\mu, A)^m = (\mu - \lambda)^m \big(\mu \mathcal{R}(\mu, A) - \mathrm{id}\big)^m \to 0 = A^m P$$

as $\mu \to \lambda$, from which the assertion follows. □

Lemma 7.3.5. *Let $(x_j)_{j\in J}$ be a net*[7] *of elements in the real part $E_{\mathbb{R}}$ of a Banach lattice E and let $u \in E$ be a quasi-interior point of E_+. Let $x \in E$ be such that x_j converges to x in the Banach lattice E_u.*

If there exists $c > 0$ such that $x \geq cu$, then for each $\varepsilon \in (0, c)$, there exists $j_0 \in J$ such that $x_j \geq \varepsilon u$ for all $j \geq j_0$.

Proof. For each $\varepsilon \in (0, c)$, we can find $j_0 \in J$ such that for each $j \geq j_0$, we have $\|x_j - x\|_u \leq c - \varepsilon$ and in turn, $|x_j - x| \leq (c-\varepsilon)u$ by the definition of gauge norm. Since each x_j is real, we obtain that $x_j \geq x - (c-\varepsilon)u \geq \varepsilon u$ for all $j \geq j_0$. □

Theorem 7.3.6. *Let $A\colon E \supseteq \operatorname{dom}(A) \to E$ be a closed, densely defined, and real operator on a complex Banach lattice E. Let $\lambda \in \sigma(A) \cap \mathbb{R}$ be a pole of the resolvent $\mathcal{R}(\cdot, A)$ and let $u \in E_+$ be a quasi-interior point. Consider the following assertions:*

(i) *The resolvent $\mathcal{R}(\cdot, A)$ is individually eventually positive with respect to u at λ.*

(ii) *The spectral projection P associated to λ satisfies $Pf \succeq u$ whenever $0 \lneq f \in E$.*

(iii) *The eigenspace $\ker(\lambda - A)$ is spanned by a vector $v \succeq u$ and $\ker(\lambda - A')$ contains a strictly positive functional ψ.*

Each of them implies that λ is algebraically simple and hence a first order pole of $\mathcal{R}(\cdot, A)$.[8] *One has (i) ⇒ (ii) ⇔ (iii), and if $\operatorname{dom}(A) \subseteq E_u$, then all three assertions are equivalent.*

Proof. *"(ii) ⇒ (iii)"*: By Proposition 7.3.2, $P = (Pu) \otimes \varphi$ for a strictly positive functional $\varphi \in E'$. In particular, $\operatorname{rg} P$ is one-dimensional.

According to Theorem 6.2.6(b) and (c), $\operatorname{rg} P$ coincides with the generalised eigenspace of λ, so it follows that λ is algebraically, and hence geometrically simple, and that $0 \lneq v \in \ker(\lambda - A) = \operatorname{rg} P$ with $v = Pv \succeq u$.

Moreover, as P is the spectral projection of A corresponding to λ, P' is the spectral projection of A' corresponding to λ (Theorem 6.2.6(d)). Thus, $\ker(\lambda - A')$ contains the strictly positive functional φ.

"(iii) ⇒ (ii)": Firstly, Lemma 7.3.3(a) ensures that λ is even algebraically simple. Therefore, we obtain from Theorem 6.2.6(b) and (c) that $\operatorname{rg} P = \ker(\lambda - A)$.

Now, if $0 \lneq f \in E$, then there exists $\alpha \in \mathbb{C}$ such that $Pf = \alpha v$. Actually, as A is real, and hence so is P, we get $\alpha \in \mathbb{R}$. We claim that $\alpha > 0$. If not, then

$$0 \lneq f \leq f - \alpha v = f - Pf \in \ker P = \operatorname{rg}(\lambda - A)$$

which implies that $\langle \psi, f - \alpha v \rangle = 0$, contradicting the strict positivity of ψ.

[7] Recall the definition of a net from Exercise 4.5.

[8] By Theorem 6.2.6(b).

"(i) ⇒ (ii)": From Theorem 6.3.3, there exist $0 \lneq v \in \ker(\lambda - A)$ and $0 \lneq \varphi \in \ker(\lambda - A')$. By assumption, there exists $\mu > \lambda$ such that $v = (\mu - \lambda)\mathcal{R}(\mu, A)v \succeq u$. In particular, v is also a quasi-interior point of E_+. Furthermore, ψ is strictly positive. Indeed, if $0 \lneq f \in E$ and choose $\mu > \lambda$ such that $\mathcal{R}(\mu, A)f \succeq u$. Then

$$\langle \psi, f\rangle = \langle (\mu - \lambda)\mathcal{R}(\mu, A')\psi, f\rangle = (\mu - \lambda)\langle \psi, \mathcal{R}(\mu, A)f\rangle \succeq \langle \psi, u\rangle > 0$$

because u is a quasi-interior (Proposition 7.1.4).

Next, let $p \in \mathbb{N}$ be the pole order of λ. Then $(\mu - \lambda)^p \mathcal{R}(\mu, A)v$ converges to Q_{-p+1} as $\mu \downarrow \lambda$ and the resolvent is individually eventually positive at λ, hence $Q_{-p+1} \geq 0$. Lemma 7.3.3(b) thus ensures that $p = 1$.

Owing to Theorem 6.2.6, $Q_0 = Q_{-p+1} \gneq 0$ is the spectral projection of A associated to λ, $\operatorname{rg} Q_0 = \ker(\lambda - A)$, and $\operatorname{rg} Q_0' = \ker(\lambda - A')$. The last equality ensures that $Q_0'\varphi = \varphi$. As φ is strictly positive, this implies $\ker Q_0$ does not contain any positive non-zero elements. So if $0 \lneq f \in E$, then $Q_0 f \gneq 0$ and in turn, there exists $\mu > \lambda$ such that

$$Q_0 f = (\mu - \lambda)\mathcal{R}(\mu, A)Q_0 f \succeq u.$$

Lastly, assume that $\operatorname{dom}(A) \subseteq E_u$.

"(ii) ⇒ (i)": As already observed, λ is a first order pole of the resolvent $\mathcal{R}(\cdot, A)$ and hence by Lemma 7.3.4, $(\mu - \lambda)\mathcal{R}(\lambda, A) \to P$ in $\mathcal{L}(E, \operatorname{dom}(A))$ as $\mu \downarrow \lambda$. Since $\operatorname{dom}(A) \subseteq E_u$, this convergence even holds in $\mathcal{L}(E, E_u)$ thanks to the closed graph theorem.

Thus for $0 \lneq f \in E$, the net $(\mu - \lambda)\mathcal{R}(\lambda, A)f$ converges to $Pf \succeq u$ in E_u. The assertion thus follows by an application of Lemma 7.3.5. □

As a natural follow-up to Theorem 7.3.6, one may ask whether the eventual positivity of the resolvent can be obtained from the spectral assertions without assuming $\operatorname{dom}(A) \subseteq E_u$. Changing the state space to $L^p(-1, 1)$ in Example 6.1.2 for $1 \leq p < \infty$ (un)fortunately, refutes this; readers interested in details of the computation can find it in [DGK16a, Example 5.4]. Let us observe next that eventual negativity also fits into the framework of Theorem 7.3.6.

Corollary 7.3.7. *In the situation of Theorem 7.3.6, assume that* $\operatorname{dom}(A) \subseteq E_u$. *Then the assertions (i)–(iii) are also equivalent to the following property.*

(iv) *The resolvent* $\mathcal{R}(\cdot, A)$ *is individually eventually negative with respect to* u *at* λ.

Proof. Without loss of generality let $\lambda = 0$. Replacing A with $-A$ in the theorem, assertion (iii) remains unchanged, but (i) becomes individual eventual positivity of $\mathcal{R}(\cdot, -A)$ with respect to u at 0, which is equivalent to (iv). □

An application of Theorem 7.3.6 and Corollary 7.3.7 to a fourth-order differential operator is discussed in Exercise 7.2. In our final example in this chapter we revisit the most prominent (and most classical) example so far, the Dirichlet Laplacian.

On bounded domains in $\mathbb{R}^n$ we already now that the Dirichlet Laplacian on L^2 has a positive eigenvector for the eigenvalue $s(\Delta_{\mathrm{Dir}})$. Using the abstract results established in the present section and the concrete results from Section 7.2 we can now show much more. In one dimension we already knew assertions (a) and (b) in the following example from a concrete computation (Example 6.3.2); such an explicit computation is, of course, not possible on general domains in dimension ≥ 2.

Example 7.3.8 (The leading eigenfunction of the Dirichlet Laplacian)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open, bounded, and connected. We assume that Ω has C^{2k} boundary for some integer $k > \frac{n}{4} + 1$. Consider the quasi-interior point u of $L^2(\Omega)_+$ given by $u(x) := \operatorname{dist}(x, \partial\Omega)$ for all $x \in \Omega$. The spectral bound $s(\Delta_{\mathrm{Dir}})$ of the Dirichlet Laplace operator $\Delta_{\mathrm{Dir}}\colon L^2(\Omega) \supseteq \operatorname{dom}(\Delta_{\mathrm{Dir}}) \to L^2(\Omega)$ has the following properties:

(a) $\ker(s(\Delta_{\mathrm{Dir}}) - \Delta_{\mathrm{Dir}})$ is spanned by a positive function v that satisfies $u \preceq v \preceq u$.

(b) One has $s(\Delta_{\mathrm{Dir}}) < 0$.[9]

(c) If $n = 1$, then $\mathcal{R}(\cdot, \Delta_{\mathrm{Dir}})$ is individually eventually negative with respect to u at $s(\Delta_{\mathrm{Dir}})$.

Proof. (a) Let us abbreviate $\lambda_0 := s(\Delta_{\mathrm{Dir}})$. Since Δ_{Dir} has compact resolvent and $-\infty < \lambda_0 \leq 0$ (Example 6.3.5), λ_0 is a pole of the resolvent and an eigenvalue (Theorem 6.2.9).

As shown in Example 7.2.4 we have $\mathcal{R}(\lambda, \Delta_{\mathrm{Dir}})f \succeq u$ for all $\lambda > \lambda_0$ and all $0 \lneq f \in L^2(\Omega)$. Hence, Theorem 7.3.6 implies that $\ker(\lambda_0 - \Delta\mathrm{Dir})$ is spanned by a function $v \succeq u$. On the other hand, for $\lambda > \lambda_0$ one has $(\lambda - \lambda_0)^{-k} v = \mathcal{R}(\lambda, \Delta_{\mathrm{Dir}})^k v$, and the latter vector is in the principal ideal E_u according to Example 7.2.4. Thus, $v \preceq u$.

(b) We use the notation from the proof of (a). Assume for a contradiction that $\lambda_0 = 0$. Then $\Delta_{\mathrm{Dir}} v = 0$. One has $v \in \operatorname{rg} \mathcal{R}(\lambda, A)^k \subseteq C^2(\overline{\Omega})$, where the inclusion was shown in Example 7.2.4. Hence, we can apply the maximum principle (Example 5.2.2) to conclude that v obtains its maximum at $\partial\Omega$. But v vanishes on $\partial\Omega$ and v is positive, so $v = 0$, a contradiction.

(c) We apply Theorem 7.3.6. As pointed out in the proof of (a), condition (i) of the theorem is satisfied. Moreover, since $n = 1$ one has

$$\operatorname{dom}(\Delta_{\mathrm{Dir}}) = H^2(\Omega) \cap H^1_0(\Omega) \subseteq C^1(\overline{\Omega}) \cap C_0(\Omega) \subseteq L^2(\Omega)_u,$$

so Corollary 7.3.7 can be applied and gives the claimed eventual negativity. □

So we have know now that $\mathcal{R}(\cdot, \Delta_{\mathrm{Dir}})$ is individually eventually negative at $s(\Delta_{\mathrm{Dir}})$ if $n = 1$ and that it is not uniformly eventually negative there when $n \geq 4$ (Example 6.4.6). This obviously leaves a gap, which we will close later on.

[9] As pointed out before, this also follows from the **Poincaré inequality** under more general assumptions on Ω, but here we give a proof based on the maximum principle.

Exercises for Chapter 7

Exercise 7.1 (Quasi-interior points)**.**

(a) Let E be a Banach lattice. Show that $u \in E_+$ is a quasi-interior point if and only if $(nu) \wedge f \to f$ as $n \to \infty$ for every $f \in E_+$.

(b) Let $p \in [1, \infty)$ and let (Ω, μ) be a σ-finite measure space.

Show that $u \in L^p(\Omega, \mu)_+$ is a quasi-interior point if and only if $u(\omega) > 0$ for almost all $\omega \in \Omega$. Also show that $v \in L^\infty(\Omega, \mu)_+$ is a quasi-interior point if and only if $v \geq \mathbb{1}$.

(c) Let K be a compact metric space. Show that $u \in \mathrm{C}(K)_+$ is a quasi-interior point if and only if $u(x) > 0$ for all $x \in K$, if and only if $u \geq \mathbb{1}$.

(d) Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be an open set. Prove that a function $u \in \mathrm{C}_0(\Omega)_+$ is a quasi-interior point if and only if $u(x) > 0$ for all $x \in \Omega$.

(e) Give an example of a Banach lattice E with no quasi-interior points.

(f) Let E be a Banach lattice and let $\varphi \in E'_+$. Which of the following conditions implies the other?

  (i) φ is strictly positive.
  (ii) φ is a quasi-interior point of E'_+.

Exercise 7.2 (A fourth order operator on an interval)**.**

(a) Consider the space $L^2(0,1)$ and endow its vector subspace

$$V := \left\{ v \in H^2(0,1) : v'(0) = v'(1) = 0 \right\}$$

with the H^2-norm. Define the sesquilinear form

$$\mathfrak{a} \colon V \times V \to \mathbb{C}, \qquad \mathfrak{a}(u,v) := \int_0^1 \overline{u''}\, v'' \,\mathrm{d}x.$$

Show that $\mathfrak{a}$ satisfies all assumptions of Theorem 5.1.4, where μ can be chosen to be any number > 0.

(b) Let $A \colon L^2(0,1) \supseteq \operatorname{dom}(A) \to L^2(0,1)$ denote the operator associated to $\mathfrak{a}$. Show that there exists $\lambda_0 > 0$ such that $\mathcal{R}(\lambda, A) \not\geq 0$ for all $\lambda \in [\lambda_0, \infty)$.

(c) Compute $\operatorname{dom}(A)$ and Au for all $u \in \operatorname{dom}(A)$.

(d) Show that $\ker A$ and $\ker(A')$ are spanned by $\mathbb{1}$, that $\operatorname{s}(A) = 0$, and that A has compact resolvent.

(e) Prove that $\mathcal{R}(\cdot, A)$ is individually eventually positive with respect to $\mathbb{1}$ at 0.

Exercise 7.3 (The positive minimum principle and higher order operators)**.** Let K be a compact metric space. Let $A\colon \mathrm{C}(K) \supseteq \operatorname{dom}(A) \to \mathrm{C}(K)$ be closed and densely defined, and assume that $\operatorname{s}(A) < \infty$ and $\mathcal{R}(\lambda, A) \geq 0$ for all $\lambda > \operatorname{s}(A)$.

(a) Show that A is real.

(b) Fix a point $\mu \in (\operatorname{s}(A), \infty)$ and set $u := \mathcal{R}(\mu, A)\,\mathbb{1}$. Prove that that u is a quasi-interior point of $\mathrm{C}(K)_+$, i.e. that $u(x) > 0$ for each $x \in K$.

Show furthermore that $\mathcal{R}(\lambda, A)u \leq \frac{1}{\lambda - \mu} u$ for all $\lambda > \mu$.

(c) Show for every $v \in \operatorname{dom}(A)$ that $\lambda\big(\lambda \mathcal{R}(\lambda, A) - \mathrm{id}\big)v \to v$ in $\mathrm{C}(K)$ as $\lambda \to \infty$.

Hint: Use Exercise 5.1(a).

(d) Prove that, for each $x_0 \in K$, the operator A satisfies the **positive minimum principle at** $x_0 \in K$, i.e. for each $0 \leq v \in \operatorname{dom}(A)$ with $v(x_0) = 0$ one has $(Av)(x_0) \geq 0$.[10]

(e) Let $m \geq 3$ be an integer. Consider a densely defined closed operator $B\colon \mathrm{C}([-1,1]) \supseteq \operatorname{dom}(B) \to \mathrm{C}([-1,1])$. Assume that $\operatorname{dom}(B)$ contains all test functions on $(-1,1)$ and that $Bv = v^{(m)}$ for each such test function v.

Show that B does not satisfy the positive minimum principle for any $x_0 \in (-1,1)$.

[10] Note that this is the same property that was assumed in Theorems 5.2.1 and 7.2.1.

Notes for Chapter 7

Quasi-interior points

Quasi-interior points were introduced by Schaefer [Sch60] as a generalisation of points in the topological interior of the cone. The main motivation was that cones in infinite-dimensional spaces often have empty interior, but quasi-interior points still have many useful properties of interior points. Quasi-interior points are, for instance, useful in the study of so-called **irreducible** operators, see e.g. [MN91, Section 4.2]. In the literature, the symbol $u \gg 0$ is sometimes used to denote that u is a quasi-interior point.

We point out that the characterisation of quasi-interior points via positive linear functionals (Proposition 7.1.4) fails in the general setting of so-called **ordered Banach spaces**. See [GW20, Section 2.2] for a detailed discussion of this topic.

The Hopf boundary point lemma

The arguments that we gave to prove the Hopf boundary lemma (Example 7.2.2) are, in principle, almost the same as one may find in standard PDE books. What is unusual about our approach is that we phrased it in the abstract setting of Theorem 7.2.1, which extends the setting of the abstract maximum principle from Theorem 5.2.1 by an additional assumption on w.

To encode the inner normal derivative $-\partial_\nu$ at x_0 into the action of the operator A in the proof of Hopf's boundary lemma does not seem to be a common approach. This somewhat unconventional structure of A, without a clear theoretical explanation for its occurrence, is one indication – among others – that the abstract versions of the maximum principle in Theorems 5.2.1 and 7.2.1 are not yet in a really satisfactory state.

Readers with an inclination towards PDE theory may also object to the strong assumptions on the regularity of the boundary of the domain Ω in Example 7.2.4. This is due to our Sobolev-space approach, which starts with very little regularity (merely L^2 functions) and heavily depends on the Sobolev embedding theorems. An equally well-established approach is the so-called **Schauder theory**, which works with spaces of Hölder continuous functions and classical derivatives. In short, if Ω is a bounded domain with $C^{2,\alpha}$ boundary (for some $\alpha \in (0,1)$), $f \in C^{0,\alpha}(\overline{\Omega})$, and if u solves $\lambda u - \Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$ in the classical sense, then $u \in C^{2,\alpha}(\overline{\Omega})$; see [GT01, Theorem 6.19]. Thus, the order of differentiability of the PDE solution and the boundary agree.

Individually eventually positive resolvents

The characterisation of individual eventual positivity with respect to u in Theorem 7.3.6 stems from [DGK16a, Sections 3 and 4]. The fact that the implication from (i) to the other assertions holds even without the domination assumption $\operatorname{dom}(A) \subseteq E_u$ was proved in [DG17, Section 4]. The proof that we presented for this implication is a bit different in that it avoids using properties of quasi-interior points that are more strongly tied to the lattice structure of the surrounding space (in particular, the properties discussed in Supplement 7.A). This might turn out beneficial in potential generalisations of eventual positivity theory to ordered Banach spaces.

The fact that the domination assumption $\operatorname{dom}(A) \subseteq E_u$ is in fact necessary in many cases in order to have individual eventual positivity and negativity at the same time, was shown in [AG23]. In these notes, we simplified the proof and improved the result by removing the assumption that the vector u satisfies $u \preceq v \preceq u$ for an eigenvector of A.

Encore: if you want to know more…

7.A More on quasi-interior points

In the Banach lattice $\mathbb{R}$, every non-zero positive element is a quasi-interior point. Here, we show that this cannot happen in any dimension larger than 1 (Proposition 7.A.4). The underlying idea is related to Proposition 7.3.2, but is technically a bit more involved.

Definition 7.A.1 (Disjointness of vectors)**.** Two vectors x, y of a Banach lattice E are called **disjoint** if $|x| \wedge |y| = 0$.

Example 7.A.2. If x is any vector in a Banach lattice E, then x^+ and x^- are disjoint elements, and thus $x = x^+ - x^-$ is a *disjoint decomposition*. Indeed, one has

$$x^+ \wedge x^- = [(x^+ - x^-) \wedge (x^- - x^-)] + x^- = (x \wedge 0) + x^- = -x^- + x^- = 0$$

by elementary properties of vector lattice operations (Proposition 4.1.3).

Lemma 7.A.3. *If E is a Banach lattice with dimension $\dim E \geq 2$, then there exist two disjoint non-zero elements $x, y \in E_+$.*

Proof. As $\dim E \geq 2$ and as the linear span of E_+ equals E, we can find two linearly independent vectors $u, v \in E_+$; in particular, $u, v \neq 0$. Observe that there exists a real number $t_0 \in (0, \infty)$ such that neither $u \leq t_0 v$ nor $u \geq t_0 v$ holds. Indeed, both the sets

$$\{t \in (0,\infty) : u \leq tv\} \quad \text{and} \quad \{t \in (0,\infty) : u \geq tv\}$$

are not equal to $(0,\infty)$ since $u, v \neq 0$, are relatively closed in $(0,\infty)$ since E_+ is closed, and are disjoint since u, v are linearly independent. Hence, the union of those two sets cannot be $(0,\infty)$ since $(0,\infty)$ is connected.

Let us now replace v with $t_0 v$. Then u, v are non-zero vectors in E_+ that satisfy $u \not\leq v$ and $u \not\geq v$. Hence, the positive vectors $x := u - u \wedge v$ and $y := v - u \wedge v$ are non-zero and they are disjoint since

$$0 \leq x \wedge y = (u - u \wedge v) \wedge (v - u \wedge v) = u \wedge v - u \wedge v = 0. \qquad \square$$

Proposition 7.A.4. *If every positive non-zero element of a Banach lattice E is a quasi-interior point, then $\dim E \leq 1$.*

Proof. Suppose $\dim E \geq 2$. By Lemma 7.A.3, there exist disjoint non-zero vectors $x, y \in E_+$. Every element of E_y is then disjoint to x: indeed, for each $z \in E_y$ there exists a number $c \geq 1$ such that $|z| \leq cy$. Hence, $x \wedge |z| = 0$ follows from

$$0 \leq x \wedge |z| \leq x \wedge (cy) \leq (cx) \wedge (cy) = c(x \wedge y) = 0,$$

From continuity of the lattice operations (Proposition 4.1.7), it follows that each element of the closure $\overline{E_y}$ is disjoint to x. Since x is non-zero, it is not disjoint to itself and thus, $x \notin \overline{E_y}$. Therefore, y is not a quasi-interior point of E_+. □

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