Chapter 6

Eventually positive resolvents and their spectral properties

We know by now that the resolvent of the Dirichlet Laplacian is positive everywhere on the right of the spectral bound (Example 5.4.2) and that this is closely related to estimates for sesquilinear forms (Exercise 4.3) and to the classical maximum principle (Example 5.3.6). The same techniques can be used to prove similar results for many second order elliptic differential operator with “nice” boundary conditions.

Therefore, the phenomenon observed at the end of the previous chapter (Example 5.4.3) might be all the more surprising: choosing slightly uncommon boundary conditions can result in the resolvent being positive in a right neighbourhood of the spectral bound, but not everywhere up to ∞. As we proceed, we shall see that this **eventual positivity** – where “eventual” means “as one moves towards the spectral bound from the right” – occurs in many more examples. Consequently, we lay the groundwork for a general theory of eventual positivity in this chapter.

6.1 Eventually positive resolvents

The following concept is at the heart of this and the next two chapters.

Definition 6.1.1 (Eventual positivity of resolvents)**.** Let $A\colon E \supseteq \operatorname{dom}(A) \to E$ be a closed operator on a complex Banach lattice E. Let $\lambda_0 \in \mathbb{R}$ be a spectral value of A such that a right neighbourhood of λ_0 is contained in $\rho(A)$. let $u \in E_+$ and $0 \le Q \in \mathcal{L}(E)$.

(a) $\mathcal{R}(\cdot\,, A)$ is said to be **individually eventually positive with respect to** u **at** λ_0 if for each $0 \lneq f \in E$ one has $\mathcal{R}(\lambda, A)f \succeq u$ for all λ in a (f-dependent) right neighbourhood of λ_0.

(b) $\mathcal{R}(\cdot\,, A)$ is said to be **uniformly eventually positive with respect to** Q **at** λ_0 if one has $\mathcal{R}(\lambda, A) \succeq Q$ for all λ in a right neighbourhood of λ_0.

The case $u \gneq 0$ and $Q \gneq 0$ in Definition 6.1.1 will become relevant in Chapter 7. In the present chapter we focus on the weakest choices of u and Q, namely $u = 0$ and $Q = 0$.

The Laplace operator with non-local boundary conditions in Example 5.4.3 is uniformly eventually positive with respect to 0 at its spectral bound, as shown in that example. It is natural to wonder whether individual and uniform eventual positivity are in fact equivalent, but here is a counterexample.

Example 6.1.2 (Individual versus uniform eventual positivity)**.** There exists an operator $A \in \mathcal{L}\big(\mathrm{C}([-1,1])\big)$ with spectrum $\sigma(A) = \{0,-1,-3\}$ and the following properties:

(a) $\mathcal{R}(\cdot, A)$ is individually eventually positive with respect to 0 at $\lambda_0 := 0$.

(b) $\mathcal{R}(\lambda, A)$ is not uniformly eventually positive with respect to 0 at $\lambda_0 = 0$.

Proof. Let $\varphi \in \mathrm{C}([-1,1])'$ be given by $\langle \varphi, f\rangle = \frac{1}{2}\int_{-1}^{1} f(\omega)\,\mathrm{d}\omega$. Let $S : \ker\varphi \to \ker\varphi$ be the reflection operator given by $(Sf)(\omega) = f(-\omega)$ for all $f \in \ker\varphi$ and all $\omega \in [-1,1]$. We define A as the block diagonal operator

$$A := \begin{pmatrix} 0 & 0 \\ 0 & -S-2 \end{pmatrix}$$

with respect to the decomposition $\mathrm{C}([-1,1]) = \mathbb{C}\,\mathbb{1} \oplus \ker\varphi$. In other words, $A\mathbb{1} = 0$ and $Af = (-S-2)f$ for all $f \in \ker\varphi$. Clearly, 1 and -1 are eigenvalues of S, so it follows from $S^2 = \mathrm{id}_{\ker\varphi}$ and the spectral mapping theorem for polynomials that $\sigma(S) = \{-1,1\}$. Hence, $\sigma(A) = \sigma(0) \cup \sigma(-S-2) = \{0,-1,-3\}$. We now prove (a) and (b).

(a) By using $S^2 = \mathrm{id}_{\ker\varphi}$ one can readily check that, for $\lambda \in \rho(A)$,

$$\mathcal{R}(\lambda, A) = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{\lambda+2-S}{(\lambda+2)^2-1} \end{pmatrix}.$$

Now let $0 \lneq f \in \mathrm{C}([-1,1])$ and write f as $f = \langle \varphi, f\rangle \mathbb{1} + g$ for a function $g \in \ker\varphi$. Of course, $\lambda\mathcal{R}(\lambda,A)f = \langle\varphi, f\rangle\mathbb{1} + \lambda\mathcal{R}(\lambda,A)g \to \langle \varphi, f\rangle \mathbb{1}$ with respect to $\|\cdot\|_\infty$ for $\lambda \downarrow 0$. Since $\lambda\mathcal{R}(\lambda,A)f$ is real for real λ and $\langle\varphi, f\rangle\mathbb{1}$ is a constant function with value $\langle \varphi, f\rangle > 0$, it follows that $\mathcal{R}(\lambda,A)f \geq 0$ for all $\lambda > 0$ that are sufficiently close to 0.

(b) Fix a number $\lambda > 0$; we show that $\mathcal{R}(\lambda, A) \ngeq 0$. For each $\varepsilon > 0$, one can choose $f_\varepsilon \in \mathrm{C}([-1,1])_+$ that satisfies $f_\varepsilon(1) = 1$, $f_\varepsilon(-1) = 0$ and $\langle \varphi, f_\varepsilon\rangle = \varepsilon$. Again, we write f_ε as $f_\varepsilon = \varepsilon\,\mathbb{1} + g_\varepsilon$ for $g_\varepsilon \in \ker\varphi$. Since $g_\varepsilon(-1) = -\varepsilon$ and $g_\varepsilon(1) = 1-\varepsilon$, we get

$$\mathcal{R}(\lambda,A)f_\varepsilon(-1) = \frac{\varepsilon}{\lambda} + \frac{(\lambda+2)g_\varepsilon(-1) - g_\varepsilon(1)}{(\lambda+2)^2-1} \xrightarrow{\varepsilon\downarrow 0} \frac{-1}{(\lambda+2)^2-1} < 0.$$

So there exists $\varepsilon > 0$ such that $\mathcal{R}(\lambda,A)f_\varepsilon(-1) < 0$. Thus $\mathcal{R}(\lambda, A) \ngeq 0$ because $f_\varepsilon \geq 0$. □

6.2 Intermezzo: Eigenvalues and poles of the resolvent

We now study eventual positivity of resolvents with similar spectral theoretic techniques as in the finite-dimensional case. In this section, we present some spectral theoretic machinery in infinite dimensions that resembles, to a certain extent, the tools from Section 2.1. For this to be possible, compactness will have to play an essential role.

In addition, we will need standard results for holomorphic functions with values in a Banach space. Readers not familiar with this theory can find a brief summary thereof in Appendix 6.A.

Proposition 6.2.1 (The resolvent is analytic)**.** *Let* $A\colon X \supseteq \operatorname{dom}(A) \to X$ *be a closed linear operator on a complex Banach space* X*. The resolvent mapping* $\mu \mapsto \mathcal{R}(\mu, A)$ *is analytic on* $\rho(A)$ *as a function with values in* $\mathcal{L}(X)$*.*

Proof. This follows from the series expansion of the resolvent in Proposition 3.3.2(a) and the characterisation of analyticity in terms of the Taylor series (Theorem 6.A.4). □

The following is an infinite-dimensional analogue of Proposition 2.1.4.

Proposition 6.2.2 (Spectral decomposition)**.** *Let* $A\colon X \supseteq \operatorname{dom}(A) \to X$ *be a closed linear operator on a complex Banach space* X *and let* $\sigma_0 \subseteq \sigma(A)$ *such that* σ_0 *is compact and* $\sigma(A) \setminus \sigma_0$ *is closed. Then there exists a unique projection* $P \in \mathcal{L}(X)$ *with the following properties:*

(a) P *commutes with* A*.*

(b) $\sigma\left(A|_{\operatorname{rg} P}\right) = \sigma_0$ *and* $\sigma\left(A|_{\ker P}\right) = \sigma(A) \setminus \sigma_0$*.*

Moreover, for any closed C^1*-cycle* γ *in* $\mathbb{C}$ *that encircles each element of* σ_0 *precisely once, but no element of* $\sigma(A) \setminus \sigma_0$*, we have*

$$P = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \mathcal{R}(\mu, A)\, \mathrm{d}\mu.$$

The notion of **cycle** that occurs in Proposition 6.2.2 means a formal sum of finitely many paths. This is necessary in infinite dimensions, since the spectrum need not consist of isolated points and hence, no single path with the required properties might exist. We do not present the proof of Proposition 6.2.2 and instead refer to the literature, e.g. [EN00, Proposition IV.1.16]. However, in the case that σ_0 is an isolated singleton which is also a pole of the resolvent, the analysis appears much like in the finite dimensional case (see Theorem 6.2.6 below).

Definition 6.2.3 (Spectral projections)**.** In the situation of Proposition 6.2.2, the projection P is called the **spectral projection** of A associated to σ_0.

After defining spectral projections in finite dimensions (Definition 2.1.5) we mentioned that the roles of σ_0 and $\sigma(A) \setminus \sigma_0$ are symmetric – i.e. swapping them gives the complementary projection. In infinite-dimensions this is, in general, only true if $A \in$

$\mathcal{L}(X)$. For unbounded A, the set $\sigma(A) \setminus \sigma_0$ can be unbounded, so that no associated spectral projection is defined. Even if the set is bounded, the spectral projections of $\sigma(A) \setminus \sigma_0$ and σ_0 need not add up to id_X – indeed, this is the case whenever $\sigma_0 := \sigma(A) = \emptyset$.

Definition 6.2.4 (Powers of unbounded operators)**.** Let $A\colon X \supseteq \operatorname{dom}(A) \to X$ be a linear operator on a Banach space X. The powers of A are defined inductively by

$$A^0 := \mathrm{id}, \quad A^{n+1}x := A(A^n x) \quad \text{for } x \in \operatorname{dom}\left(A^{n+1}\right) := \left\{x \in \operatorname{dom}\left(A^n\right) : A^n x \in \operatorname{dom}(A)\right\}.$$

Of course, $\operatorname{dom}(A^n)$ is a decreasing sequence of subspaces. Moreover, $A^m A^n = A^{m+n}$ whenever the composition is defined. Exercise 6.2 explores such properties.

Definition 6.2.5 ((Generalised) eigenspaces and semisimplicity)**.** Let $A\colon X \supseteq \operatorname{dom}(A) \to X$ be a linear operator on a complex Banach space X and let $\lambda \in \mathbb{C}$.

(a) λ is called an **eigenvalue of** A if $\ker(\lambda - A) \neq \{0\}$. In this case, $\ker(\lambda - A)$ is called the associated **eigenspace** and its non-zero elements are called the associated **eigenvectors**. The dimension of $\ker(\lambda - A)$ is called the **geometric multiplicity** of λ.

The **point spectrum** $\sigma_{\mathrm{pnt}}(A)$ is the set of all eigenvalues of A.

(b) If λ is an eigenvalue of A, then the subspace $\bigcup_{k\in\mathbb{N}} \ker(\lambda - A)^n$ of $\operatorname{dom}(A)$ is called the associated **generalised eigenspace**. Its dimension is called the **algebraic multiplicity** of λ.

(c) If λ is an eigenvalue of A, it is called a **semisimple eigenvalue** if its generalised eigenspace coincides with its eigenspace.

Note that $\ker(\lambda - A) \subseteq \ker(\lambda - A)^2 \subseteq \dots$, so the generalised eigenspace is indeed a subspace of $\operatorname{dom}(A)$. In finite dimensions, semisimplicity of an eigenvalue λ is obviously equivalent to equality of its geometric and algebraic multiplicities. However, in infinite dimensions, it can happen that both multiplicities are ∞ without equality of the eigenspace and the generalised eigenspace.

For a spectral value that is a pole of the resolvent, many of the properties from the finite-dimensional setting (Proposition 2.1.6) remain true. We summarise the most important ones for our purposes in the following theorem.

Theorem 6.2.6 (Poles of the resolvent)**.** *Let A be a closed operator on a complex Banach space X and let $\lambda \in \sigma(A)$ be a pole of the resolvent $\mathcal{R}(\cdot, A) : \rho(A) \to \mathcal{L}(X)$ of order $p \in \mathbb{N}$. Let*

$$\mathcal{R}(\mu, A) = \sum_{k=-p}^{\infty} Q_{k+1}(\mu - \lambda)^k.$$

denote the Laurent series expansion of the resolvent about λ with coefficients $Q_{k+1} \in \mathcal{L}(X)$.[1]

(a) *One has $\{0\} \neq \operatorname{rg} Q_{-p+1} \subseteq \ker(\lambda - A)$. In particular, λ is an eigenvalue of A.*

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[1] Note that $Q_{-p+1} \neq 0$ since p is the pole order of λ.

(b) *One has* $\operatorname{rg} Q_0 = \ker(\lambda - A)^k$ *for all* $k \geq p$*, so* $\operatorname{rg} Q_0$ *is the generalised eigenspace of* A *for the eigenvalue* λ*. Thus, the eigenvalue* λ *is semisimple if and only if* $p = 1$.

(c) *The coefficient* $Q_0 \in \mathcal{L}(X)$ *is the spectral projection of* A *associated with* λ.

(d) *If* A *is densely defined, then* λ *is also a pole of* $\mathcal{R}(\cdot, A')$ *of order* p *with the Laurent series expansion*

$$\mathcal{R}(\mu, A') = \sum_{n=-p}^{\infty} Q'_{k+1}(\mu - \lambda)^k.$$

In particular, λ *is also an eigenvalue of* A' *with the associated spectral projection* Q'_0.

Proof of (c) *and* (d). (c) As in Proposition 2.1.6(a), this follows from contour integral formula of the spectral projection in Proposition 6.2.2 and the scalar-valued integral formulas in Proposition 2.1.2.

(d) This follows from (a) by using the fact $\mathcal{R}(\cdot, A') = \mathcal{R}(\cdot, A)'$ from Exercise 3.5(d). ☐

The proofs of (a) and (b) rely on a detailed analysis of the coefficients Q_k of the Laurent series expansion. To stay on track, we again refrain from discussing those arguments here. Readers fond of complex analysis or looking for a deeper understanding of spectral theory, can find the full analysis in Supplement 6.B in Theorems 6.B.1 and 6.B.3.

In practice, a very convenient way to check that a spectral value is a pole of the resolvent is to use the following concept and Theorem 6.2.9 below.

Definition 6.2.7 (Compact resolvent)**.** A closed operator A on a complex Banach space X is said to have **compact resolvent** if there exists $\lambda \in \rho(A)$ such that $\mathcal{R}(\lambda, A) : X \to X$ is a compact operator.

Proposition 6.2.8. *Let* $A \colon X \supseteq \operatorname{dom}(A) \to X$ *be a closed operator on a complex Banach space* X*. The following are equivalent:*

(i) *A has compact resolvent.*

(ii) $\rho(A) \neq \emptyset$ *and the operator* $\mathcal{R}(\lambda, A) \colon X \to X$ *is compact for each* $\lambda \in \rho(A)$.

(iii) $\rho(A) \neq \emptyset$ *and embedding* $\operatorname{dom}(A) \hookrightarrow X$ *is compact.*

Proof. "(i) ⇒ (iii)": Let $\lambda \in \rho(A)$ be such that $\mathcal{R}(\lambda, A) \colon X \to X$ is compact. Composing with $\lambda - A \in \mathcal{L}(\operatorname{dom}(A), X)$ yields that $\operatorname{id}_{\operatorname{dom}(A)} \colon \operatorname{dom}(A) \to X$ is compact.

"(iii) ⇒ (ii)": Let $\lambda \in \rho(A)$. If the embedding $\operatorname{dom}(A) \hookrightarrow X$ is compact, then its composition with $\mathcal{R}(\lambda, A) \in \mathcal{L}(X, \operatorname{dom}(A))$ ensures that $\mathcal{R}(\lambda, A) \colon X \to X$ is compact.

"(ii) ⇒ (i)": This implication is obvious. ☐

It can happen that $\operatorname{dom}(A)$ embeds compactly into X but $\rho(A) = \emptyset$. For instance, this is the case for the first order differential operator on $\mathrm{C}([0,1])$ from Example 3.3.4(a), where the compact embedding follows from the Arzelà–Ascoli theorem.

In examples, compactness of the resolvent is often established by using a compact embedding result for Sobolev spaces to check condition (iii) in Proposition 6.2.8. The reason why we are interested in operators with compact resolvent is that they have particularly nice spectral properties that are reminiscent of the finite-dimensional case.

Theorem 6.2.9 (Compact resolvent and spectrum)**.** *Let $A\colon X \supseteq \operatorname{dom}(A) \to X$ be a closed operator on a complex Banach space X. If A has compact resolvent, then*

(a) *$\sigma(A)$ consists only of eigenvalues.*

In fact, each spectral value of A is a pole of the resolvent and an eigenvalue of finite algebraic multiplicity.

(b) *$\sigma(A)$ has no accumulation points in $\mathbb{C}$.*

One can either derive Theorem 6.2.9 from the Riesz–Schauder spectral theory of compact operators by using that the resolvent satisfies appropriate spectral mapping results, or one can obtain it as a special case of the so-called **analytic Fredholm theory**, see e.g. [GGK90, Section XI.8]. Theorem 6.2.9 will only be useful for our purposes if there are any spectral values at all.[2] For operators associated to symmetric sesquilinear forms, this is guaranteed by part (c) of the following result.

Proposition 6.2.10 (Compact embedding of form domains)**.** *Let the Hilbert spaces V, H, the sesquilinear form $\mathfrak{a}\colon V \times V \to \mathbb{C}$, and its associated operator $A\colon H \supseteq \operatorname{dom}(A) \to H$ satisfy the assumptions of Theorem 5.1.4. Then:*

(a) *The inclusion map $\operatorname{dom}(A) \hookrightarrow V$ is continuous.*

(b) *If the embedding $V \hookrightarrow H$ is compact, then A has compact resolvent.*

(c) *If the form $\mathfrak{a}$ is symmetric, then $\sigma(A) \neq \emptyset$.*[3]

Proof of (a) *and* (b). (a) Recall that $\operatorname{dom}(A) \subseteq V \subseteq H$ and that all three spaces are Banach spaces. The embeddings $\operatorname{dom}(A) \hookrightarrow H$ and $V \hookrightarrow H$ are continuous, so the closed graph theorem implies that the inclusion map from $\operatorname{dom}(A)$ into V is also continuous (Exercise 3.3(a)).

(b) Now assume that $V \hookrightarrow H$ is compact. Then it follows from (a) that $\operatorname{dom}(A) \hookrightarrow H$ is compact. Moreover, one has $\rho(A) \neq \emptyset$ by Theorem 5.1.4(b). ☐

We will not prove part (c) in the main text, but the interested reader can find a proof in Theorem 6.C.5(a) in Supplement 6.C.

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[2] Note that there are operators with compact resolvent and empty spectrum (Example 3.3.4(b)).

[3] Readers familiar with form methods might recognise that this is a special case of the fact that self-adjoint operators have non-empty spectrum.

6.3 Spectral consequences of eventual positivity

To apply the spectral theoretic results from the previous sections in concrete examples, one needs tools to check that an operator has compact resolvent. Compact embedding theorems for Sobolev spaces are very useful for this purpose. The following result in one dimension is easy to prove, given the properties from Theorem 5.3.7.

Theorem 6.3.1. *Let $\emptyset \neq I \subset \mathbb{R}$ be an open bounded interval. For all $1 < p \leq \infty$, the embeddings $W^{1,p}(I) \hookrightarrow \mathrm{C}(\bar{I})$ and $W^{1,p}(I) \hookrightarrow L^p(I)$ are compact.*

Proof. Let $B = \{u \in W^{1,p}(I) : \|u\|_{W^{1,p}} \leq 1\}$ be the closed unit ball in $W^{1,p}(I)$. By the embedding $W^{1,p}(I) \hookrightarrow \mathrm{C}(\bar{I})$ from Theorem 5.3.7(b), we deduce that B is bounded in $\mathrm{C}(\bar{I})$. By the fundamental theorem of calculus for Sobolev functions (Theorem 5.3.7(a)) and Hölder's inequality, it follows for all $u \in B$ that

$$\left|u(x) - u(y)\right| \leq \int_y^x \left|u'(t)\right| \,\mathrm{d}t \leq \left\|u'\right\|_{L^p(I)} \left|x-y\right|^{1-1/p} \qquad \forall\, x, y \in \bar{I}. \tag{6.3.1}$$

Since $\left\|u'\right\|_{L^p(I)} \leq 1$ for all $u \in B$ and $p > 1$, the above inequality shows that B is equicontinuous. Hence, the Arzelà-Ascoli theorem implies that B is compact in $\mathrm{C}(\bar{I})$. Finally, the embedding $\mathrm{C}(\bar{I}) \hookrightarrow L^p(I)$ is continuous, and the remaining assertion follows. ☐

The case $p = 1$ is different and will be treated in Exercise 6.5. Here is a first example of how compactness of the resolvent can be used to determine the spectrum of an operator.

Example 6.3.2. Consider the Dirichlet Laplacian $\Delta_{\mathrm{Dir}} \colon L^2(0,\pi) \supseteq \operatorname{dom}(\Delta_{\mathrm{Dir}}) \to L^2(0,\pi)$.

(a) One has $\sigma(\Delta_{\mathrm{Dir}}) = \sigma_{\mathrm{pnt}}(\Delta_{\mathrm{Dir}}) = \left\{-k^2 : k \in \mathbb{N}\right\}$.

(b) The eigenspace $\ker(-k^2 - \Delta_{\mathrm{Dir}})$ is spanned by $\sin(k\,\cdot)$ for all $k \in \mathbb{N}$.

In particular, $\mathrm{s}(\Delta_{\mathrm{Dir}}) = -1$ is an eigenvalue with a positive eigenvector.

Proof. One readily checks for every $k \in \mathbb{N}$ that $\sin(k\,\cdot)$ is indeed in $\operatorname{dom}(\Delta_{\mathrm{Dir}})$ and is an eigenvector of Δ_{Dir} for the eigenvalue $-k^2$. Let us show that, conversely, every spectral value is an eigenvalue of the form $-k^2$ and that its eigenspace is spanned by $\sin(k\,\cdot)$.

First, recall from Exercise 3.3.6(b) that Δ_{Dir} is the operator associated to a sesquilinear form $\mathfrak{a}$ on $H_0^1(0,\pi)$ and that $\mathfrak{a}$ satisfies the assumptions of Theorem 5.1.4. The embedding $H_0^1(0,\pi) \hookrightarrow L^2(0,\pi)$ is compact (Theorem 6.3.1) and thus, Δ_{Dir} has compact resolvent (Proposition 6.2.10).

We already know that $\sigma(\Delta_{\mathrm{Dir}}) \subseteq (-\infty, 0]$ (Example 5.4.2) and the compactness of the resolvent implies that every spectral value is actually an eigenvalue (Theorem 6.2.9(a)). Now, let $\lambda \in (-\infty, 0]$ be an eigenvalue of Δ_{Dir} with a corresponding eigenvector $0 \neq u \in \operatorname{dom}(\Delta_{\mathrm{Dir}}) \subseteq H^2(0,\pi)$. Then $u'' = \Delta_{\mathrm{Dir}} u = \lambda u \in H^2(0,\pi)$. A simple inductive argument[4] yields $u \in H^k(0,\pi) \subseteq \mathrm{C}^{k-1}([0,\pi])$ for all $k \in \mathbb{N}$, where the latter inclusion is due to the one-dimensional Sobolev embedding theorem 5.3.7(b).

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[4] This is often called **bootstrapping** in the PDE literature.

Classical ODE theory can thus be applied to $u'' = \lambda u$ and yields that $u = \alpha \cos(\sqrt{-\lambda}\,\cdot) + \beta \sin(\sqrt{-\lambda}\,\cdot)$ for suitable scalars $\alpha, \beta \in \mathbb{C}$. As $u \in H_0^1(\Omega)$ one has $u(0) = u(\pi) = 0$ (Exercise 6.3). The equality $u(0) = 0$ implies $\alpha = 0$, so $0 \neq u = \beta \sin(\sqrt{-\lambda}\,\cdot)$. The equality $u(\pi) = 0$ now gives $\sqrt{-\lambda} \in \mathbb{N}$, i.e. there exists $k \in \mathbb{N}$ such that $\lambda = -k^2$ and $u = \beta \sin(k\,\cdot)$. ☐

We already know from Example 5.4.2 that the resolvent $\mathcal{R}(\cdot, \Delta_{\mathrm{Dir}})$ is positive on the right of the spectral bound $\mathrm{s}(\Delta_{\mathrm{Dir}})$. In Example 6.3.2 we have now seen by a direct computation that the eigenvalue $\mathrm{s}(\Delta_{\mathrm{Dir}})$ has a positive eigenvector. Thinking back to the Perron–Frobenius type results in finite dimensions (Theorems 1.3.8 and 1.3.9 in the positive case, Theorems 2.2.3 and 2.3.1 in the eventually positive case), one might wonder whether the existence of a positive eigenvector is a consequence of (eventual) positivity of the resolvent in general infinite-dimensional situations, as well. The answer is 'yes':

Theorem 6.3.3 (Eigenvectors for eventually positive resolvents)**.** *Let A be a closed operator on a complex Banach lattice E and let $\lambda \in \mathbb{R}$ be a pole of the resolvent $\mathcal{R}(\cdot, A)$.*[5] *If $\mathcal{R}(\cdot, A)$ is individually eventually positive with respect to* 0 *at λ, then:*

(a) *There exists a positive eigenvector $v \in E$ of A for the eigenvalue λ.*

(b) *If the eigenvalue λ is semisimple, then its associated spectral projection is positive.*

(c) *If A is densely defined, then A' has a positive eigenvector $\varphi \in E'$ for the eigenvalue λ.*

Proof. Let $p \in \mathbb{N}$ denote the pole order of λ and consider the the Laurent series expansion

$$\mathcal{R}(\mu, A) = \sum_{k=-p}^{\infty} Q_{k+1}(\mu - \lambda)^k$$

of the resolvent about λ, where $Q_{-p+1} \neq 0$ and where the range of Q_{-p+1} is contained in the eigenspace $\ker(\lambda - A)$ according to Theorem 6.2.6(a). One has $Q_{-p+1} = \lim_{\mu \to \lambda} (\mu - \lambda)^p \mathcal{R}(\mu, A)$. For every $x \in E_+$ it thus follows that $Q_{-p+1} = \lim_{\mu \downarrow \lambda} (\mu - \lambda)^p \mathcal{R}(\mu, A) x \geq 0$ due to the eventual positivity of $\mathcal{R}(\cdot, A)$. So Q_{-p+1} is a positive operator.

(a) Since E_+ spans E and $Q_{-p+1} \neq 0$ there exists a vector $x \in E_+$ such that $v := Q_{-p+1} x \neq 0$. Hence, v is a positive eigenvector of A for the eigenvalue λ.

(b) If the eigenvalue λ is semisimple, then $p = 1$ and $0 \leq Q_{-p+1} = Q_0$ is the spectral projection (Theorem 6.2.6(b) and (c)).

(c) By Theorem 6.2.6(a) and (d), λ is an eigenvalue of A' and $\mathrm{rg}(Q'_{-p+1}) \subseteq \ker(\lambda - A')$. As the dual cone E'_+ spans E' and $Q'_{-p+1} \neq 0$, there exists $\psi \in E'_+$ such that $0 \neq \varphi := Q'_{-p+1}\psi \in \ker(\lambda - A')$. Also $\varphi \geq 0$ because $Q'_{-p+1} \geq 0$ (Corollary 4.4.5). ☐

Let us demonstrate Theorem 6.3.3 in two examples.

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[5] Hence, λ is an eigenvalue of A and, if A is densely defined, also an eigenvalue of A' (Theorem 6.2.6(a) and (d)).

Example 6.3.4. Let $\Delta_B \colon L^2(0,1) \supseteq \operatorname{dom}(\Delta_B) \to L^2(0,1)$ denote the Laplace operator with non-local boundary conditions from Example 5.4.3. Then $-\infty < \operatorname{s}(\Delta_B) < 0$ and $\operatorname{s}(\Delta_B)$ is an eigenvalue of Δ_B with a positive eigenvector.

Proof. We know from Example 5.4.3 that $\operatorname{s}(\Delta_B) < 0$ and $\mathcal{R}(\lambda, \Delta_B) \geq 0$ for all $\lambda \in (\operatorname{s}(\Delta_B), 0]$. Moreover, Proposition 6.2.10(b) shows that Δ_B has compact resolvent since the embedding of the associated form domain $H^1(0,1)$ into $L^2(0,1)$ is compact (Theorem 6.3.1). Thus, all spectral values of Δ_B are poles of the resolvent (Theorem 6.2.9(a)).

As the form that we used to define Δ_B is symmetric, one has $\sigma(\Delta_B) \neq \emptyset$ (Proposition 6.2.10(c)), equivalently $\operatorname{s}(\Delta_B) > -\infty$. As $\sigma(\Delta_B) \subseteq (-\infty, 0)$ (Example 5.4.3) is non-empty and closed, $\operatorname{s}(\Delta_B) \in \sigma(\Delta_B)$ and hence is a pole of $\mathcal{R}(\cdot, \Delta_B)$. So we can apply Theorem 6.3.3(a) to see that there exists a positive eigenvector for $\operatorname{s}(\Delta_B)$. ☐

In contrast to the simple case on the interval that was treated in Example 6.3.2, for the Dirichlet Laplacian on general domains in $\mathbb{R}^n$, the eigenvalues or eigenvectors cannot be explicitly computed. Yet a positive eigenvector corresponding to $\operatorname{s}(\Delta_{\mathrm{Dir}})$ exists:

Example 6.3.5 (Dirichlet Laplacian on domains)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open and bounded and consider the Dirichlet Laplacian Δ_{Dir} on $L^2(\Omega)$. Then the spectral bound $\operatorname{s}(\Delta_{\mathrm{Dir}})$ is not $-\infty$ and it is an eigenvalue with a positive eigenvector.

Proof. The argument is the same as in Example 6.3.4, except that one now has to use compactness of Sobolev embeddings in several dimensions (Theorem 6.3.6 below). ☐

The following result is a version of Theorem 6.3.1 in several dimensions. It was used in the previous example and will be frequently useful in the rest of the course.

Theorem 6.3.6 (Compact embeddings of Sobolev spaces)**.** *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open and bounded and let $p \in [1, \infty)$.*

(a) *The embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*

(b) *If Ω has C^1-boundary, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*

The proof of Theorem 6.3.6 usually relies on the Fréchet-Kolmogorov compactness theorem (see Exercise 6.5). Details may be found in [Bre11, Theorem 9.16].

6.4 The left neighbourhood of spectral values

For an operator A, eventual positivity of $\mathcal{R}(\cdot, A)$ at a point $\lambda_0 \in \mathbb{R}$ means, loosely speaking, that the solution u to $(\lambda - A)u = f$ is positive if $f \geq 0$ and λ is in a right neighbourhood of λ_0. It is natural to ask what happens on the left of λ_0. A glance at the case $E = \mathbb{C}$ suggests that one might expect negativity of solutions there. It turns out that this is the correct idea in principle, but there are also quite a few subtleties and surprises.

Definition 6.4.1 (Eventual negativity of resolvents)**.** Let $A \colon E \supseteq \operatorname{dom}(A) \to E$ be a closed operator on a complex Banach lattice E. Let $\lambda_0 \in \mathbb{R}$ be a spectral value of A such that a left neighbourhood of λ_0 is contained in $\rho(A)$. let $u \in E_+$ and $0 \leq Q \in \mathcal{L}(E)$.

(a) $\mathcal{R}(\cdot, A)$ is said to be **individually eventually negative with respect to u at λ_0** if for each $0 \lneq f \in E$, one has $\mathcal{R}(\lambda, A) f \preceq -u$ for all λ in an (f-dependent) left neighbourhood of λ_0.

(b) $\mathcal{R}(\cdot, A)$ is said to be **uniformly eventually negative with respect to Q at λ_0** if one has $\mathcal{R}(\lambda, A) \preceq -Q$ for all λ in a left neighbourhood of λ_0.

We will see in the subsequent chapters that eventual positivity indeed occurs in various examples. In the present section we focus on the opposite phenomenon: a rather surprising result that says that, in many cases, eventual positivity and negativity are not possible at the same time. We need the following notion from Banach lattice theory.

Definition 6.4.2 (Ideals in vector lattices)**.** Let E be a real vector lattice or a (real or complex) Banach lattice.[6] An **ideal** in E is a vector subspace $I \subseteq E$ such that, for all $x, y \in E$, the properties $y \in I$ and $|x| \le |y|$ imply that $x \in I$.

Note that a vector subspace $I \subseteq E$ is an ideal if and only if I is a vector sublattice and $0 \le x \le y \in I$ implies $x \in I$. Let us give examples of ideals in our favourite Banach lattices.

Examples 6.4.3.

(a) Let (Ω, μ) be a σ-finite measure space and let $\Omega' \subseteq \Omega$ be measurable. For $p \in [1, \infty]$, it is easy to check that

$$I_{\Omega'} := \left\{ f \in L^p(\Omega, \mu) : f|_{\Omega'} = 0 \right\}$$

is a closed ideal of $L^p(\Omega, \mu)$. If $p \neq \infty$ one can prove that, in fact, all closed ideals in $L^p(\Omega, \mu)$ are of this form [BKFR17, Propositions 10.15].

(b) Let (Ω, μ) be a σ-finite measure space. Clearly, $L^\infty(\Omega, \mu)$ is an ideal in $L^p(\Omega, \mu)$ for each $p \in [1, \infty]$. It is not closed unless $p = \infty$ or $L^p(\Omega, \mu)$ is finite-dimensional.

(c) Let K be a compact metric space [7] and let $J \subseteq K$ be a closed set. Then

$$I_J := \left\{ f \in \mathrm{C}(K) : f|_J = 0 \right\}$$

is a closed ideal of $\mathrm{C}(K)$. Again, one can prove that all closed ideals in $\mathrm{C}(K)$ are of this form [BKFR17, Propositions 10.13].

(d) Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be an open set, and $p \in [1, \infty)$. Then $W_0^{1,p}(\Omega; \mathbb{R})$ is an ideal in $W^{1,p}(\Omega; \mathbb{R})$. A proof for the case $p = 2$, which can easily be adapted for general $p \in [1, \infty)$, can be found in [AU23, Theorem 6.39]. An important ingredient is the fact that the lattice operations are continuous on $W^{1,p}(\Omega)$, cf. Remark 4.B.4.

In many examples, one can use the following theorem to exclude eventually negative behaviour. The assumption $\operatorname{dom}(A^m) \subseteq I$ and the conclusion $\operatorname{dom}(A) \subseteq I$ is closely related to Sobolev embedding theorems, as we shall see in Example 6.4.6.

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[6] This slightly strange assumption is merely due to the fact that we did not define complex vector lattices.

[7] Or more generally, a compact Hausdorff topological space.

Theorem 6.4.4. *Let* $A\colon E \supseteq \operatorname{dom}(A) \to E$ *be a closed operator with compact resolvent on a complex Banach lattice* E *and let* $\lambda_0 \in \mathbb{R}$ *be an isolated spectral value of* A. *Let* $I \subseteq E$ *be an ideal (not necessarily closed) and assume:*

(1) $\operatorname{dom}(A^m) \subseteq I$ *for some* $m \in \mathbb{N}$.

(2) $\mathcal{R}(\,\cdot\,, A)$ *is uniformly eventually positive and negative with respect to* 0 *at* λ_0.

Then $\operatorname{dom}(A) \subseteq I$.

In Chapter 7 we will show that the theorem remains true, up to minor changes, if one only assumes individual eventual positivity and negativity in (2); but this requires a bit more preparation. In the rest of Chapter 6 we give the rather simple proof of Theorem 6.4.4 and discuss an example. We use the following finite expansion of the resolvent.

Lemma 6.4.5. *Let* A *be a closed operator on a complex Banach space* X *and* $n \in \mathbb{N}_0$. *Then*

$$\mathcal{R}(\lambda, A) = \sum_{k=1}^{n} (\mu-\lambda)^{k-1}\mathcal{R}(\mu, A)^k + \mathcal{R}(\lambda, A)(\mu-\lambda)^n\mathcal{R}(\mu, A)^n \qquad \textit{for all } \lambda, \mu \in \rho(A).$$

Proof. This follows by iterating the resolvent identity (Proposition 3.3.2(c)). ☐

Proof of Theorem 6.4.4. By assumption (2) there are $\lambda, \mu \in \rho(A) \in \mathbb{R}$ that satisfy $\lambda < \lambda_0 < \mu$ such that $\mathcal{R}(\lambda, A) \le 0$ and $\mathcal{R}(\mu, A) \ge 0$. Abbreviating $S := \mathcal{R}(\lambda, A)(\mu-\lambda)^{m-1}\mathcal{R}(\mu, A)^{m-1}$, the finite resolvent expansion in Lemma 6.4.5 gives

$$0 \le -\mathcal{R}(\lambda, A) = -\sum_{k=1}^{m-1} (\mu-\lambda)^{k-1}\mathcal{R}(\mu, A)^k \;-\; S \le -S.$$

For every $f \in E_+$ we conclude that $0 \le -\mathcal{R}(\lambda, A)f \le -Sf \in I$, since S maps into $\operatorname{dom}(A^m)$ and thus, by assumption (1), into I. As I is an ideal in E, it follows that $\mathcal{R}(\lambda, A)f \in I$. Since E_+ spans E, this shows that $\operatorname{dom}(A) = \operatorname{rg}\mathcal{R}(\lambda, A) \subseteq I$, as claimed. ☐

Example 6.4.6. Let $\emptyset \ne \Omega \subseteq \mathbb{R}^n$ be open and bounded with C^2 boundary and consider the Dirichlet Laplacian Δ_{Dir} on $L^2(\Omega)$. If $n \ge 4$, the resolvent $\mathcal{R}(\,\cdot\,, \Delta_{\mathrm{Dir}})$ is not uniformly eventually negative with respect to 0 at $\mathrm{s}(\Delta_{\mathrm{Dir}})$.

Proof. We apply the contrapositive of Theorem 6.4.4, for the ideal $I := L^\infty(\Omega)$ in $L^2(\Omega)$. Assumption (1) of the theorem is satisfied: by iterating Theorem 5.3.2(b) one obtains $\operatorname{dom}\left(\Delta_{\mathrm{Dir}}^m\right) \subseteq H^{2m}(\Omega)$ for each $m \in \mathbb{N}$, so the Sobolev embedding theorem 5.3.4 ensures that $\operatorname{dom}\left(\Delta_{\mathrm{Dir}}^m\right) \subseteq L^\infty(\Omega)$ whenever $n < 4m$. Moreover, the first part of assumption (2) holds since $\mathcal{R}(\,\cdot\,, \Delta_{\mathrm{Dir}})$ is uniformly eventually positive with respect to 0 at $\mathrm{s}(\Delta_{\mathrm{Dir}})$ by Example 5.4.2.

To see that $\mathcal{R}(\,\cdot\,, \Delta_{\mathrm{Dir}})$ is not uniformly eventually negative with respect to 0 at $\mathrm{s}(\Delta_{\mathrm{Dir}})$, we observe that the conclusion of theorem is not satisfied. Indeed, it is a classical fact that $H^2(\Omega) \cap H^1_0(\Omega)$ does not embed into $L^\infty(\Omega)$ for $n \ge 4$, i.e. $\operatorname{dom}(\Delta_{\mathrm{Dir}}) \not\subseteq L^\infty(\Omega)$; for instance, the famous example mentioned in Remark 5.B.5(a) can be modified to ensure that the Dirichlet boundary conditions are satisfied. ☐

Exercises for Chapter 6

Exercise 6.1. Give an example of a closed operator A on a complex Banach space X and a set $\emptyset \neq \sigma_0 \subsetneq \sigma(A)$ such that σ_0 and $\sigma(A) \setminus \sigma_0$ are both compact yet the spectral projections corresponding to σ_0 and $\sigma(A) \setminus \sigma_0$ do not add up to id_X.

Exercise 6.2. Let $A\colon X \supseteq \operatorname{dom}(A) \to X$ be an operator on a Banach space X.

(a) Let $m, n \in \mathbb{N}_0$ and $x \in \operatorname{dom}\left(A^{m+n}\right)$. Show that $x \in \operatorname{dom}(A^n)$, $A^n x \in \operatorname{dom}(A^m)$, and

$$A^m(A^n x) = A^{m+n} x.$$

(b) Let $\lambda \in \mathbb{C}$. Show that $\operatorname{dom}((\lambda - A)^n) = \operatorname{dom}(A^n)$ for all $n \in \mathbb{N}$.

Give an example of

(c) an operator A on a Banach space X such that $\operatorname{dom}\left(A^2\right) \neq \operatorname{dom}(A)$;

(d) a closed operator A on a Banach space X such that A^2 is not closed.

Exercise 6.3 (A characterisation of $W_0^{1,p}(I)$)**.** Let $p \in [1, \infty)$ and let $a, b \in \mathbb{R}$, $a < b$. In this exercise we show that

$$W_0^{1,p}(a,b) = \left\{u \in W^{1,p}(a,b) : u(a) = u(b) = 0\right\}$$

This is the one-dimensional version of a result that we have mentioned multiple times.

(a) Show the inclusion "$\subseteq$" using the continuity of the embedding $W^{1,p}(a,b) \hookrightarrow \mathrm{C}([a,b])$.

(b) Show the inclusion "$\supseteq$". You may use that there is a function $w \in \mathrm{C}^\infty([a,b])$ that is constantly 0 in some right neighbourhood of a and constantly 1 in some left neighbourhood of b.

Hint: For $u \in W^{1,p}(a,b)$ with $u(a) = u(b) = 0$, take test functions φ_n that converge in L^p to u'. Then consider the functions $(1 - w)u_n + w v_n$, where

$$u_n(x) := \int_a^x \varphi_n(s) \,\mathrm{d}s \qquad \text{and} \qquad v_n(x) := \int_b^x \varphi_n(s) \,\mathrm{d}s$$

for all $x \in (a,b)$.

Exercise 6.4. Consider the operator $A\colon \mathrm{C}([0,1]) \supseteq \operatorname{dom}(\to) \mathrm{C}([0,1])$ given by

$$\begin{aligned} \operatorname{dom}(A) &:= \{u \in \mathrm{C}^1([0,1]) : u'(0) = u'(1)\}, \\ Au &= u'. \end{aligned}$$

(a) Prove that A is closed, densely defined, and has compact resolvent.[8]

(b) Compute all eigenvalues of A.

(c) Compute the eigenspace $\ker A$ and the generalised eigenspace $\bigcup_{n\in\mathbb{N}} \ker A^n$. Show that both spaces are spanned by positive vectors.

(d) Compute the eigenspace $\ker A'$ and the generalised eigenspace $\bigcup_{n\in\mathbb{N}} \ker (A')^n$ of the dual operator A' on $\mathrm{C}([0,1])'$ (Definition 3.1.5).

Hint: First use Theorem 6.2.6(b) and (d) to determine $\dim \bigcup_{n\in\mathbb{N}} \ker (A')^n$.

(e) Is $\mathcal{R}(\cdot, A)$ individually eventually positive with respect to 0 at the spectral value 0? Is it individually eventually negative with respect to 0 at 0?

Exercise 6.5.

(a) Show that the embedding $W^{1,1}(0,1) \hookrightarrow \mathrm{C}([0,1])$ is not compact.

(b) Show that the embedding $W^{1,1}(0,1) \hookrightarrow L^1(0,1)$ is compact.

Hint: By the Fréchet–Kolmogorov compactness theorem, a subset $F \subseteq L^1(0,1)$ is relatively compact if and only if

$$\sup_{f\in F} \int_{(0,1)} \left| f(s+h) - f(s) \right| \,\mathrm{d}s \to 0$$

as $h \to \infty$. Here, one extends each f by 0 outside of $(0,1)$ to always make sense of the integral.

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[8] Beware that it does not suffice for the compactness of the resolvent to note that the embedding $\mathrm{C}^1([0,1]) \hookrightarrow \mathrm{C}([0,1])$ is compact.

Notes for Chapter 6

Eventual positivity and negativity and the (anti-)maximum principle

As demonstrated in Example 5.3.6, positivity of $\mathcal{R}(\cdot\,, \Delta_{\mathrm{Dir}})$ is closely related to the classical maximum principle. Inspired by this link – which holds also for more general second order operators with a variety of boundary conditions – some authors simply use the term **maximum principle** to refer to what we call eventual positivity of the resolvent (or versions thereof). Consequently, eventual negativity of the resolvent is then referred to as an **anti-maximum principle**, for instance by Clément & Sweers [CS00, CS01] and Grunau & Sweers [SG01]. An abstract operator theoretic approach to anti-maximum principles is due to Takáč [Tak96].

The notion **eventually positive resolvent** was used in [DGK16b, DGK16a], where it mainly served as a tool to better understand eventually positive operator semigroups – a topic that we will discuss in later chapters. Example 6.1.2 is taken from [DGK16b, Example 5.7]. A similar example where the operator has compact resolvent can be found in [DGK16b, Example 5.8]. Theorem 6.3.3 is, up to a few modifications, taken from [DG17, Theorem 3.1], which in turn generalised results from [DGK16a]. The idea for Theorem 6.4.4 stems from [AG22, AG23], although we now presented it in a somewhat different perspective. It is remarkable that the positivity and negativity of solutions to the equation $(\lambda - A)u = f$ for λ in a neighbourhood of the spectral value λ_0, together with the a priori regularity assumption $\operatorname{dom}(A^m) \subseteq I$, leads to improved regularity of such solutions, which is captured by the property $\operatorname{dom}(A) \subseteq I$.

Appendices

6.A Vector-valued analytic functions

This appendix collects some basic facts on complex-analytic mappings taking values in Banach spaces. We will mainly apply the concepts and results from this appendix to the resolvents of linear operators, which are analytic maps according to Proposition 6.2.1.

Definition 6.A.1. Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open and let X be a complex Banach space. A function $f : \Omega \to X$ is called **analytic** (or **holomorphic**) if

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists in X for each $z_0 \in \Omega$.

Note that analytic implies continuous and in fact, as in the scalar-valued case, if f is analytic, then so is f'. Hence, one can iteratively define the the k-th derivatives $f^{(k)}$ for $k \in \mathbb{N}_0$ (with the usual convention $f^{(0)} := f$). Moreover, if $f : \Omega \to X$ is analytic, then it is also **weakly analytic**, i.e. $x' \circ f$ is analytic for each $x' \in X'$ with

$$(x' \circ f)'(z_0) = \langle x', f'(z_0) \rangle .$$

It turns out that analyticity and weak analyticity are equivalent notions. In fact, even more is true (Theorem 6.A.6).

As in the scalar-valued case, contour integrals are an important concept for vector-valued analytic functions. They are defined in precisely the same way, where the occurring integral is a Bochner integral. To describe spectral projections in full generality, one needs the contour integral not only over closed C^1-curves, but also over formal sums thereof – these are called C^1**-cycles**. For a C^1-cycle γ and a point $z_0 \in \mathbb{C}$ we write, by slight abuse of notation, $z_0 \not\in \gamma$ to say that z_0 does not lie in the image of any of the curves of γ.

A contour integral over a cycle γ is naturally defined as the sum over the contour integrals of all the curves it contains. The **winding number** of γ around a point $z_0 \not\in \gamma$ is thus also defined. We say that γ **encircles** z_0 **once** if this winding number is 1 and we say that γ **does not encircle** z_0 if the winding number if 0.

Proposition 6.A.2 (Cauchy's integral theorem for vector-valued functions)**.** *Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open, let X be a complex Banach space, and let $f : \Omega \to X$ be analytic. Let γ be a C^1-cycle in Ω that does not encircle any point in $\mathbb{C} \setminus \Omega$.*

(a) *One has* $\oint_\gamma f(z)\,\mathrm{d}z = 0$.

(b) *If* $z_0 \in \Omega$ *satisfies* $z_0 \notin \gamma$ *and is encircled once by* γ*, then*

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z.$$

The identity theorem for vector-valued analytic functions can be worded as follows.

Proposition 6.A.3 (Identity theorem for analytic functions)**.** *Let* $\emptyset \neq \Omega \subseteq \mathbb{C}$ *be open and connected,* X *be a Banach space,* $Y \subseteq X$ *a closed vector subspace, and let* $f : \Omega \to X$ *be analytic. Let* $(z_n) \subseteq \Omega$ *be a convergent sequence such that* $(f(z_n)) \subseteq Y$*. If* $\lim_{n\to\infty} z_n \in \Omega$*, then* $f(z) \in Y$ *for all* $z \in \Omega$*.*

This follows from the scalar valued identity theorem by testing against elements of X'. Note that the 'usual' statement of the identity theorem is recovered by taking $Y = \{0\}$.

Analogous to the scalar-valued case, we obtain Taylor and Laurent series expansions for vector-valued analytic functions.

Theorem 6.A.4 (Taylor expansion)**.** *Let* $\emptyset \neq \Omega \subseteq \mathbb{C}$ *be open let* X *be a complex Banach space, and let* $f : \Omega \to X$*. The following are equivalent:*

(i) *The function* f *is analytic on* Ω*.*

(ii) *For each* $z_0 \in \mathbb{C}$*, there exist* $(a_k) \subseteq X$ *and radius* $r \in (0, \operatorname{dist}(z_0, \partial\Omega))$ *such that* $f(z) = \sum_{k=0}^\infty a_{k+1}(z - z_0)^k$ *for all* $z \in B_{\leq r}(z_0)$*, with absolute uniform convergence on this disk.*

(iii) *For each* $z_0 \in \mathbb{C}$ *there exists* $(a_k) \subseteq X$ *such that* $f(z) = \sum_{k=0}^\infty a_{k+1}(z - z_0)^k$ *for all* $z \in B_{<\operatorname{dist}(z_0,\partial\Omega)}(z_0)$*, with absolute uniform convergence on compact subsets of this disk.*

If these equivalent conditions are satisfied, the coefficients $a_1, a_2, \dots$ *in* (ii) *and* (iii) *do not depend on the choice of* r *and are given by*

$$a_{k+1} = \frac{1}{2\pi \mathrm{i}} \oint_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{k+1}}\,\mathrm{d}z = \frac{f^{(k)}(z_0)}{k!}$$

for all $k \in \mathbb{N}_0$ *and any* $0 < r < \operatorname{dist}(z_0, \partial\Omega)$*.*

Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open, let X be a complex Banach space and let $f : \Omega\setminus \to X$ be analytic. As in the scalar-valued case, a point $z_0 \in \mathbb{C}$ is called a **pole** of f if z_0 is an isolated point of $\mathbb{C} \setminus \Omega$ and there exists a $p \in \mathbb{N}$ such that $\lim_{z\to z_0} (z - z_o)^p f(z)$ exists (in norm) and is non-zero. In this case, p is uniquely determined and is called the **order** of the pole z_0.

Theorem 6.A.5 (Laurent expansion)**.** *Let* $\emptyset \neq \Omega \subseteq \mathbb{C}$ *be open, let* X *be a complex Banach space, let* $z_0 \in \Omega$ *and let* $f : \Omega \setminus \{z_0\} \to X$ *be analytic.*

(a) *Then there exists $r > 0$ such that $B_{\le r}(z_0) \setminus \{z_0\} \subseteq \Omega$*

$$f(z) = \sum_{k=-\infty}^{\infty} a_{k+1}(z - z_0)^k \qquad \text{for all } z \in B_{\le r}(z_0) \setminus \{z_0\}$$

with absolute uniform convergence on compact subsets of the disk, where

$$a_{k+1} = \frac{1}{2\pi \mathrm{i}} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{k+1}} \,\mathrm{d}z \qquad \text{for all } k \in \mathbb{Z}.$$

(b) *z_0 is a pole of f if and only if there exists $p \in \mathbb{N}$ such that the coefficients a_{k+1} from (a) satisfy $a_{-p+1} \neq 0$ and $a_{-k+1} = 0$ for all $k > p$. In this case, p is the order of the pole.*

Finally, we expand a bit more on the connection between analyticity and weak analyticity. We call a function $f\colon M \to X$ from a metric space M into a Banach space X **locally bounded** if every point $z_0 \in M$ has a neighbourhood U such that $\sup_{z \in U} \|f(z)\| < \infty$. Recall that a vector subspace Y of a dual Banach space X' is weak*-dense in X' if and only if it is **separating**, i.e. $\langle x', x\rangle = 0$ for a vector $x \in X$ and all $x' \in Y$ implies $x = 0$.

Theorem 6.A.6. *Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open, let X be a Banach space, and let $f\colon \Omega \to X$ be locally bounded. The following are equivalent.*

(i) *f is analytic.*

(ii) *$x' \circ f$ is analytic for all x' in a weak* -dense subspace of X.*

Note that if the condition (ii) in the theorem is satisfied for all $x \in X$, then the local boundedness of f follows automatically from the uniform boundedness theorem. For the proof we refer for instance to [ABHN11, Theorem A.7].

Corollary 6.A.7. *Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open, let X, Y be Banach spaces, and let $f : \Omega \to \mathcal{L}(X,Y)$ be locally bounded. The following are equivalent.*

(i) *f is analytic.*

(ii) *$f(\cdot)x$ is analytic for all x in a dense subspace of X.*

(iii) *$\langle y', f(\cdot)x\rangle$ is analytic for all x in a dense subspace of X and all y' in a weak* -dense subspace of Y'.*

As before, we note that if condition (ii) in the corollary is satisfied for all $x \in X$, or if (iii) is satisfied for all $x \in X$ and for all $y' \in Y$, then the local boundedness assumption on f is automatically satisfied due to the uniform boundedness principle.

Encore: if you want to know more…

6.B Isolated singularities of the resolvent

In this supplemental section, we expand a bit on the background of Theorem 6.2.6. The essence of the ideas is in the following Theorem 6.B.1, which analyses the coefficients of the Laurent series expansion of the resolvent around an isolated spectral value. Thus, the missing parts of the proof of Theorem 6.2.6 are then deduced in Theorem 6.B.3 below.

For a closed linear operator $A : X \supseteq \operatorname{dom}(A) \to X$ on a complex Banach space X, we use the notation $\operatorname{dom}(A^\infty) := \bigcap_{j \in \mathbb{N}_0} \operatorname{dom}\left(A^j\right)$.

Theorem 6.B.1 (Isolated singularities of the resolvent)**.** *Let $A : X \supseteq \operatorname{dom}(A) \to X$ be a closed linear operator on a complex Banach space X and let $\lambda \in \sigma(A)$ be an isolated point in $\sigma(A)$. Let*

$$\mathcal{R}(\mu, A) = \sum_{k=-\infty}^{\infty} Q_{k+1} (\mu - \lambda)^k.$$

denote the Laurent series expansion of the resolvent about λ. Then the operators $Q_k \in \mathcal{L}(X)$ commute, and the following assertions hold:

(a) *For each $k \in \mathbb{Z}$ one has $\operatorname{rg} Q_k \subseteq \operatorname{dom}(A)$ and $Q_k A x = A Q_k x$ for all $x \in \operatorname{dom}(A)$.*

(b) *For all $k \in \mathbb{N}$ one has*

$$Q_k = (-1)^{k+1} Q_1^k \qquad \text{and} \qquad Q_{-k} = (Q_{-1})^k.$$

(c) *$Q_k Q_{-\ell} = 0$ for all $k, \ell \in \mathbb{N}$.*

(d) *Q_0 is a projection and satisfies*

$$Q_0 Q_k = 0 \qquad \text{and} \qquad Q_0 Q_{-k} = Q_{-k} \qquad \text{for all } k \in \mathbb{N}.$$

(e) *For every $k \in \mathbb{Z}$ one has*

$$(\lambda - A) Q_k = \begin{cases} -Q_{k-1}, & \text{if } k \neq 1, \\ \operatorname{id} - Q_0, & \text{if } k = 1. \end{cases}$$

(f) *The spectral radius $\operatorname{r}(Q_{-1})$ is 0, and hence $\operatorname{r}(Q_{-k}) = 0$ for all $k \in \mathbb{N}$.*

(g) *If $q \geq 1$ and $Q_{-q} = 0$, then λ is a pole of the resolvent, and its pole order is at most q.*

(h) *If $d := \dim(\operatorname{rg} Q_0) < \infty$, then λ is a pole of the resolvent of order d.*

(i) $\operatorname{rg} Q_{-k} \subseteq \operatorname{dom}(A^\infty)$ *for all $k \in \mathbb{N}_0$.*

Proof. First, we observe, according to Theorem 6.A.5(a), that

$$Q_k = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{\mathcal{R}(\mu, A)}{(\mu - \lambda)^k} \, \mathrm{d}\mu \tag{6.B.1}$$

for all $k \in \mathbb{Z}$, where γ denotes any sufficiently small circle about λ which is oriented anticlockwise. Since the resolvent operators of A all commute, this readily implies that the Q_k mutually commute.

(a) Endow $\operatorname{dom}(A)$ with a graph norm $\| \cdot \|_A$. This renders $\operatorname{dom}(A)$ a Banach space, since A is closed.

For every $\lambda \in \rho(A)$, the operator $\mathcal{R}(\lambda, A) : X \to \operatorname{dom}(A)$ is continuous by the closed graph theorem. Moreover, the mapping $\mathcal{R}(\cdot, A) : \rho(A) \to \mathcal{L}(X, \operatorname{dom}(A))$ is continuous; this follows from the preceding sentence together with the fact that the mapping is continuous with values in $\mathcal{L}(X)$ and the resolvent identity. Consequently, by Example 4.A.9, the integrand in (6.B.1) is Bochner integrable[9] with values in $\mathcal{L}(X, \operatorname{dom}(A))$. As the latter space embeds continuously into $\mathcal{L}(X)$, it follows that the integrals in both spaces coincide. Hence, $Q_k \in \mathcal{L}(X, \operatorname{dom}(A))$ and $Q_k X \subseteq \operatorname{dom}(A)$ for each $k \in \mathbb{Z}$.

For every $x \in \operatorname{dom}(A)$ and every $\mu \in \rho(A)$, one has $A\mathcal{R}(\mu, A)x = \mathcal{R}(\mu, A)Ax$. By applying the equality (6.B.1), and together with the facts that the integral in this equality can be interpreted as a Riemann integral in $\mathcal{L}(X, \operatorname{dom}(A))$ and that A is continuous from $\operatorname{dom}(A)$ to X, we thus obtain $Q_k A x = A Q_k x$ for every $k \in \mathbb{Z}$.

(b), (c), *and* (d) Let $k_1, k_2 \in \mathbb{Z}$ and consider formula (6.B.1) for two sufficiently small concentric circles γ_1, γ_2 with centre λ, where γ_2 has larger radius than γ_1. Then

$$\begin{aligned}
Q_{k_2} Q_{k_1} &= \frac{1}{(2\pi \mathrm{i})^2} \oint_{\gamma_2} \oint_{\gamma_1} \frac{\mathcal{R}(\mu_2, A)\mathcal{R}(\mu_1, A)}{(\mu_2 - \lambda)^{k_2} (\mu_1 - \lambda)^{k_1}} \, \mathrm{d}\mu_1 \, \mathrm{d}\mu_2 \\
&= \frac{1}{(2\pi \mathrm{i})^2} \oint_{\gamma_2} \oint_{\gamma_1} \frac{\mathcal{R}(\mu_2, A) - \mathcal{R}(\mu_1, A)}{(\mu_1 - \mu_2)(\mu_2 - \lambda)^{k_2} (\mu_1 - \lambda)^{k_1}} \, \mathrm{d}\mu_1 \, \mathrm{d}\mu_2 \\
&= \frac{1}{(2\pi \mathrm{i})^2} \oint_{\gamma_2} \frac{\mathcal{R}(\mu_2, A)}{(\mu_2 - \lambda)^{k_2}} \oint_{\gamma_1} \frac{1}{(\mu_1 - \mu_2)(\mu_1 - \lambda)^{k_1}} \, \mathrm{d}\mu_1 \, \mathrm{d}\mu_2 \\
&\quad - \frac{1}{(2\pi \mathrm{i})^2} \oint_{\gamma_1} \frac{\mathcal{R}(\mu_1, A)}{(\mu_1 - \lambda)^{k_1}} \oint_{\gamma_2} \frac{1}{(\mu_1 - \mu_2)(\mu_2 - \lambda)^{k_2}} \, \mathrm{d}\mu_2 \, \mathrm{d}\mu_1
\end{aligned}$$

By employing the residue theorem, one can compute the above integrals (where one has to distinguish several cases based on the signs of k_1 and k_2) and thus obtain the formulas claimed in (b), (c), and (d). We omit the computations.

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[9] In fact, the integral also exists as a Riemann integral.

(e) Fix $k \in \mathbb{Z}$. For every $\mu \in \rho(A)$ one has $(\lambda - A)\mathcal{R}(\mu, A) = (\lambda - \mu)\mathcal{R}(\mu, A) + \mathrm{id}$. Since the integral in the formula (6.B.1) can be interpreted as a Riemann integral in $\mathcal{L}(X, \operatorname{dom}(A))$ and since $\lambda - A : \operatorname{dom}(A) \to X$ is continuous, it follows that

$$\begin{aligned}(\lambda - A)Q_k &= \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{(\lambda - \mu)\mathcal{R}(\mu, A) + \mathrm{id}}{(\mu - \lambda)^k} \,\mathrm{d}\lambda \\ &= \frac{1}{2\pi \mathrm{i}} \oint_\gamma - \frac{\mathcal{R}(\mu, A)}{(\mu - \lambda)^{k-1}} \,\mathrm{d}\lambda + \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{\mathrm{id}}{(\mu - \lambda)^k} \,\mathrm{d}\mu.\end{aligned}$$

The first summand above is $-Q_{k-1}$, and the second summand is equal to 0 if $k \neq 1$, and equal to id if $k = 1$.

(f) Let $\varepsilon > 0$ be sufficiently small, such that (6.B.1) holds for the circle γ with radius ε about λ. For every integer $k \geq 1$ it follows from (b) that

$$\left\| Q_{-1}^k \right\| = \| Q_{-k} \| \leq \frac{1}{2\pi} \oint_\gamma \frac{\left\| \mathcal{R}(\mu, A) \right\|}{\varepsilon^{-k}} \,\mathrm{d}|\mu| \leq \sup_{\lambda \in \gamma} \left\| \mathcal{R}(\mu, A) \right\| \varepsilon^{k+1}.$$

We now use the spectral radius formula: by taking the k-th root and letting $k \to \infty$, we thus see that $\mathrm{r}(Q_{-1}) \leq \varepsilon$. This shows that $\mathrm{r}(Q_{-1}) = 0$, as claimed.

As $Q_{-k} = Q_{-1}^k$ for all $k \geq 1$, it follows from the spectral mapping theorem for polynomials that $\mathrm{r}(Q_{-k}) = 0$ for all $k \geq 1$ as well.

(g) If $Q_{-q} = 0$, then it follows from (b) that $Q_{-(q+j)} = Q_{-q}Q_{-j} = 0$ for all $j \geq 0$ as well; this shows the claim.

(h) The operator Q_{-1} commutes with Q_0, so it leaves the range of Q_0 invariant. Moreover, Q_{-1} has spectral radius 0, so its restriction to $\operatorname{rg} Q_0$ is nilpotent; more precisely, the d-th power of this restriction is 0. Hence, $Q_{-d} = Q_{-1}^d = Q_{-1}^d Q_0 = 0$, so according to (g), λ is indeed a pole of order at most d.

(i) Fix an integer $k \leq 0$. We show by induction over n that $Q_{-k}X \subseteq \operatorname{dom}(A^n)$ for each $n \in \mathbb{N}$. In (a) we proved the claim for $n = 1$, so assume now that the claim holds for some $n \in \mathbb{N}$. Let $x \in X$. It follows from (d) that $Q_{-k} = Q_{-k}Q_0$, and from (a) that $\operatorname{rg} Q_0 \subseteq \operatorname{dom}(A)$. Hence,

$$A^n Q_{-k} x \in A^n Q_{-k}(\operatorname{rg} Q_0) = A^{n-1} Q_{-k} A(\operatorname{rg} Q_0),$$

where we used the formula from (a) for the equality (which is possible since $\operatorname{rg} Q_0 \subseteq \operatorname{dom}(A)$). Since $Q_{-k}X \subseteq \operatorname{dom}(A^n)$ by the induction hypothesis, it follows that one has $A^{n-1} Q_{-k} A(\operatorname{rg} Q_0) \subseteq \operatorname{dom}(A)$. Thus we have shown that $A^n Q_{-k} x \in \operatorname{dom}(A)$, which implies $Q_{-k} x \in \operatorname{dom}\left(A^{n+1}\right)$ as claimed. □

Note that property (b) in Theorem 6.B.1 implies that λ is a pole of $\mathcal{R}(\cdot, A)$ if and only if Q_{-1} is nilpotent. In this case, the pole order is the smallest integer $q \geq 1$ such that $(Q_{-1})^q = 0$.

Remark 6.B.2 (The action of Q_{-1})**.** In the situation of Theorem 6.B.1 it follows from (d) and (e) that

$$Q_{-1}Q_0 = Q_{-1} = (A-\lambda)Q_0.$$

In other words, on $\operatorname{rg} Q_0$ the operator Q_{-1} acts as the operator $A-\lambda$.

Let us now demonstrate how Theorem 6.2.6 in the main text follows from Theorem 6.B.1. Assertions (c) and (d) of Theorem 6.2.6 have already been shown in the main text, so we focus on (a) and (b) here. For easier reference from within the proof, we state those two parts of the theorem here again.

Theorem 6.B.3 (Poles of the resolvent)**.** *Let* $A\colon X \supseteq \operatorname{dom}(A) \to X$ *be a closed operator on a complex Banach space* X *and let* $\lambda \in \sigma(A)$ *be a pole of the resolvent* $\mathcal{R}(\cdot, A) : \rho(A) \to \mathcal{L}(X)$ *of order* $p \in \mathbb{N}$. *Let*

$$\mathcal{R}(\mu, A) = \sum_{k=-p}^{\infty} Q_{k+1}(\mu-\lambda)^k.$$

denote the Laurent series expansion of the resolvent about λ *with coefficients* $Q_{k+1} \in \mathcal{L}(X)$ *and with* $Q_{-p+1} \neq 0$.

- (a) *One has* $\{0\} \neq \operatorname{rg} Q_{-p+1} \subseteq \ker(\lambda - A)$. *In particular,* λ *is an eigenvalue of* A.
- (b) *One has* $\operatorname{rg} Q_0 = \ker(\lambda-A)^k$ *for all* $k \geq p$, *so* $\operatorname{rg} Q_0$ *is the generalised eigenspace of* A *for the eigenvalue* λ. *Thus, the eigenvalue* λ *is semisimple if and only if* $p = 1$.

Proof. (a) Since Q_{-p+1} is non-zero, so is its range. Moreover, Q_{-p+1} maps into $\operatorname{dom}(A)$ according to Theorem 6.B.1(e), and part (e) of the same theorem shows that $(\lambda - A)Q_{-p+1} = -Q_{-p} = 0$.

(b) Fix $k \geq q$. It follows from Theorem 6.B.1(e) that

$$(\lambda - A)^k Q_0 = (-1)^k Q_{-k} = 0,$$

so $\operatorname{rg} Q_0 \subseteq \ker\big((\lambda-A)^k\big)$. Conversely, let $x \in \ker\big((\lambda-A)^k\big)$. Since Q_0 is a projection, so is $\operatorname{id} - Q_0$, and hence it follows from Theorem 6.B.1(e) that

$$(\operatorname{id} - Q_0)x = (\operatorname{id} - Q_0)^k x = Q_1^k(\lambda - A)^k x = 0.$$

This proves that $x \in \operatorname{rg} Q_0$.[10] □

A classical example of an operator with an isolated spectral value that is not a pole of the resolvent, is the Volterra operator:

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[10] Note that this argument could just as well be used to show directly that $\ker\big((\lambda-A)^k\big) \subseteq \operatorname{rg} Q_0$ for every $k \geq 1$ rather than just for every $k \geq q$. However, this is not important here, since the spaces $\ker\big((\lambda-A)^k\big)$ are increasing with respect to k anyway.

Example 6.B.4 (An essential singularity of the resolvent)**.** Consider the **Volterra operator**, $A\colon \mathrm{C}([0,1]) \to \mathrm{C}([0,1])$ defined by

$$Af(x) := \int_0^x f(t)\,\mathrm{d}t, \qquad x \in [0,1].$$

It is straightforward to show by induction and Fubini's theorem that

$$A^n f(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)\,\mathrm{d}t, \qquad x \in [0,1].$$

This formula easily yields the operator norm estimate

$$\left\| A^n \right\|_{\mathcal{L}(\mathrm{C}([0,1]))} \le \frac{1}{n!} \qquad \forall\, n \in \mathbb{N}.$$

Since $(n!)^{1/n} \to \infty$, the spectral radius formula yields

$$\mathrm{r}(A) = \lim_{n\to\infty} \left\| A^n \right\|^{1/n} \le \lim_{n\to\infty} \frac{1}{(n!)^{1/n}} = 0,$$

and thus we have shown that $\sigma(A) = \{0\}$.

The spectral value 0 is not an eigenvalue. Indeed, the operator A is injective: if $Af = 0$ for some $f \in \mathrm{C}([0,1])$, then we may differentiate the equation to find that $f = 0$. However, 0 is an essential singularity of the resolvent of A. To see this, we observe that $\lambda - A$ is invertible for every $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mathcal{R}(\lambda, A)$ is represented by the Neumann series:

$$\mathcal{R}(\lambda, A) = \lambda^{-1}(\mathrm{id} - \lambda^{-1}A)^{-1} = \sum_{k=0}^{\infty} A^k \lambda^{-(k+1)}.$$

Therefore, in the notation of Theorem 6.B.1, we have $Q_{-k} = A^k$ for all $k \in \mathbb{N}_0$, and the Laurent expansion of $\mathcal{R}(\lambda, A)$ around 0 has infinitely many non-zero terms in its singular part.

6.C Positivity of leading eigenvectors via sesquilinear forms

We have seen in Section 5.1 that sesquilinear forms are quite useful for constructing linear operators and establishing properties of their spectrum. Moreover, the Beurling–Deny criterion (Theorem 5.1.7) shows that they can be used to characterise positivity of the resolvent of an operator everywhere on the right of its spectral bound. In this supplement, we show how this line of thought can be developed further, in particular for symmetric forms. We first establish how the **numerical range** of a form is related to the spectral bound of the associated operator A. Then we show that for symmetric forms which satisfy the Beurling–Deny criterion, positivity of a leading eigenvector can also be established directly by form methods as an alternative to Theorem 6.3.3.

Definition 6.C.1 (Numerical range of a sesquilinear form)**.** Let V be a complex Hilbert space and $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a sesquilinear form. The set

$$\mathrm{W}(\mathfrak{a}) := \left\{\mathfrak{a}(u,u) : u \in V,\ \|u\|_V = 1\right\} \subseteq \mathbb{C}$$

is called the **numerical range of** $\mathfrak{a}$.

Recall again from Theorem 5.1.4(c) that the form $\mathfrak{a}$ is called **symmetric** if $\mathfrak{a}(u,v) = \overline{\mathfrak{a}(v,u)}$ for all $u, v \in V$. Let us note that $\mathfrak{a}$ is symmetric if and only if $\mathrm{W}(\mathfrak{a}) \subseteq \mathbb{R}$. Indeed, the symmetry implies that $\mathfrak{a}(u,u) = \overline{\mathfrak{a}(u,u)}$ and thus $\mathfrak{a}(u,u) \in \mathbb{R}$ for all $u \in V$. The converse implication follows from the polarisation identity for sesquilinear forms.

We will now prove a variety of results under the following general assumptions.

Setting 6.C.2. Let V, H be complex Hilbert spaces with the dense embedding $V \hookrightarrow H$. Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a bounded sesquilinear form which satisfies the ellipticity estimate

$$\operatorname{Re}\mathfrak{a}(u,u) + \mu\,\|u\|_H^2 \ge \delta\,\|u\|_V^2 \qquad \forall\, u \in V \tag{6.C.1}$$

for some $\mu \in \mathbb{R}$ and $\delta > 0$. Denote by $A\colon H \supseteq \operatorname{dom}(A) \to H$ the associated operator.

Lemma 6.C.3. *In Setting 6.C.2, assume that* $\operatorname{Re}\mathfrak{a}(u,u) \ge 0$ *for all* $u \in V$ *– i.e. the numerical range* $\mathrm{W}(\mathfrak{a})$ *is contained in the closed right half plane. Then for every* $\varepsilon > 0$*, there exists* $\delta_\varepsilon > 0$ *such that*

$$\operatorname{Re}\mathfrak{a}(u,u) + \varepsilon\,\|u\|_H^2 \ge \delta_\varepsilon\,\|u\|_V^2 \qquad \forall\, u \in V.$$

Proof. We proceed by contradiction. Assume that there exists $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$, there exists $u_n \in V$ with $\|u_n\|_V = 1$ such that

$$\operatorname{Re}\mathfrak{a}(u_n,u_n) + \varepsilon_0\,\|u_n\|_H^2 < \frac{1}{n} \qquad \forall\, n \in \mathbb{N}.$$

Since $\operatorname{Re}\mathfrak{a}(u_n,u_n) \ge 0$, it follows that $\|u_n\|_H \to 0$ and $\operatorname{Re}\mathfrak{a}(u_n,u_n) \to 0$ as $n \to \infty$. This contradicts the ellipticity estimate (6.C.1) because $\|u_n\|_V = 1$ for all n. ☐

Proposition 6.C.4. *In Setting 6.C.2, if* $\operatorname{Re}\mathfrak{a}(u,u) \ge 0$ *for all* $u \in V$*, then* $\mathrm{s}(A) \le 0$.

Proof. For every $\varepsilon > 0$, it follows from Lemma 6.C.3 and Theorem 5.1.4(b) that $\mathrm{s}(A) \le \varepsilon$, and thus $\mathrm{s}(A) \le 0$. ☐

The first part of the following theorem yields a proof for Proposition 6.2.10(c).

Theorem 6.C.5. *In Setting 6.C.2, assume that* $\mathfrak{a}$ *is symmetric (and hence* $\sigma(A) \subseteq \mathbb{R}$ *by Theorem 5.1.4(c)). Then the following assertions hold:*

(a) $\mathrm{s}(A) = -\inf \mathrm{W}(\mathfrak{a}) \in \sigma(A)$. *In particular,* $\sigma(A) \neq \emptyset$.

(b) *Let* $v \in V$. *Then* $\mathfrak{a}(v,v) = \inf \mathrm{W}(\mathfrak{a})$ *if and only if* $\inf \mathrm{W}(\mathfrak{a})$ *is an eigenvalue of* A *and* v *is a corresponding eigenvector.*[11]

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[11] And hence, in particular, that $v \in \operatorname{dom}(A)$.

(c) *Every eigenvalue of A is semisimple.*

In the proof we use that if a sesquilinear form $\mathfrak{a}\colon V \times V \to \mathbb{C}$ is symmetric and satisfies $\mathfrak{a}(u,u) \geq 0$ for all $u \in V$ – equivalenty, $\mathrm{W}(\mathfrak{a}) \subseteq [0,\infty)$ –, then it satisfies the Cauchy–Schwarz inequality

$$|\mathfrak{a}(u,v)| \leq \mathfrak{a}(u,u)^{1/2}\mathfrak{a}(v,v)^{1/2}$$

for all $u, v \in V$. The proof is the same as for inner products.

Proof of Theorem 6.C.5. (a) We assume without loss of generality that $\inf \mathrm{W}(\mathfrak{a}) = 0$. As shown in Proposition 6.C.4, one then has $\mathrm{s}(A) \leq 0$, so it remains to show that $\mathrm{s}(A)$ is a spectral value. Assume to the contrary that $0 \in \rho(A)$.

As $\inf \mathrm{W}(\mathfrak{a}) = 0$, we can find a sequence (v_n) in V such that $\|v_n\|_V = 1$ for all n and $\mathfrak{a}(v_n, v_n) \to 0$. First we note that, since $\mathfrak{a}$ is bounded, there exists a $c \geq 0$ such that, for all $n \in \mathbb{N}$, one has

$$\mathfrak{a}\big(\mathcal{R}(0,A)v_n, \mathcal{R}(0,A)v_n\big)^{1/2} \leq c\,\|\mathcal{R}(0,A)v_n\|_V \leq c\,\|\mathcal{R}(0,A)\|_{\mathcal{L}(V)};$$

here we used that $\mathcal{R}(0,A)$ is bounded from H to $\operatorname{dom}(A)$ and thus, in particular, from V to V since $\operatorname{dom}(A)$ embeds continuously into V by the closed graph theorem. Thus,

$$\begin{aligned}\|v_n\|_H^2 &= -(v_n \mid A\mathcal{R}(0,A)v_n))_H = \mathfrak{a}(v_n, \mathcal{R}(0,A)v_n) \\ &\leq \mathfrak{a}(v_n,v_n)^{1/2}\mathfrak{a}\big(\mathcal{R}(0,A)v_n, \mathcal{R}(0,A)v_n\big)^{1/2} \leq \mathfrak{a}(v_n,v_n)^{1/2} c\,\|\mathcal{R}(0,A)\|_{\mathcal{L}(V)}\end{aligned}$$

where the first inequality uses the Cauchy–Schwarz inequality mentioned before the proof. Hence, $\mathfrak{a}(v_n,v_n) \to 0$ implies that $\|v_n\|_H^2 \to 0$. The ellipticity estimate in Setting 6.C.2 thus shows that $\|v_n\|_V \to 0$, which is absurd.

(b) As in (a), we assume without loss of generality that $\inf \mathrm{W}(\mathfrak{a}) = 0$.

If $v \in \operatorname{dom}(A)$ and $Av = 0$, then $\mathfrak{a}(v,v) = -(v \mid Av)_H = 0$. Now assume conversely that $\mathfrak{a}(v,v) = 0$, and let $w \in V$ be arbitrary. Again by the Cauchy–Schwarz inequality, we have

$$|\mathfrak{a}(w,v)| \leq \mathfrak{a}(w,w)^{1/2}\mathfrak{a}(v,v)^{1/2} = 0.$$

Hence $\mathfrak{a}(w,v) = 0$ for all $w \in V$, which implies $v \in \operatorname{dom}(A)$ and $Av = 0$.

(c) Let $\lambda \in \mathbb{R}$ be an eigenvalue of A. Now we shift the form so that, without loss of generality, $\lambda = 0$. Let $v \in \ker A^2$. It suffices to show that $v \in \ker A$. One has

$$\|Av\|_H^2 = (Av \mid Av)_H = \mathfrak{a}(Av, v) = \mathfrak{a}(v, Av) = \big(v \mid A^2 v\big)_H = 0,$$

where the penultimate equality uses that $Av \in \operatorname{dom}(A)$ and hence $Av \in V$. Thus $Av = 0$, as claimed. ☐

The assertion of Theorem 6.C.5(b) can be rephrased by saying that a non-zero vector $v \in V$ is in $\ker(\mathrm{s}(A) - A)$ if and only if it minimises the **Rayleigh quotient**, a nonlinear functional on V defined by

$$Q(v) := \frac{\mathfrak{a}(v,v)}{\|v\|_V^2}.$$

Now we give a form based proof for the existence of a positive eigenvector for $\mathrm{s}(A)$ that was promised at the beginning of this supplemental section.

Theorem 6.C.6. *In Setting 6.C.2, assume that $H = L^2(\Omega,\mu)$ for a σ-finite measure space (Ω,μ). Let $\mathfrak{a}$ be real and symmetric and assume that $\mathcal{R}(\lambda, A) \geq 0$ for all sufficiently large $\lambda > \mathrm{s}(A)$.*[12] *If $\mathrm{s}(A)$ is an eigenvalue of A, then it has a positive eigenvector.*

Proof. Without loss of generality, we assume $\mathrm{s}(A) = 0$ and hence, $\inf \mathrm{W}(\mathfrak{a}) = 0$ by Theorem 6.C.5(b). It follows from the Beurling–Deny criterion (Theorem 5.1.7) that the real part $V_{\mathbb{R}} := V \cap L^2(\Omega,\mu;\mathbb{R})$ of the form domain V is a vector sublattice of $L^2(\Omega,\mu;\mathbb{R})$ and that $\mathfrak{a}(v^-, v^+) \leq 0$ for all $v \in V_{\mathbb{R}}$.

Now let $0 \neq v \in \ker A$, and write $v = v_1 + \mathrm{i}v_2$ for real-valued functions v_1, v_2. Since the form $\mathfrak{a}$ is real, so is the operator A (Proposition 5.1.6) and hence, $v_1, v_2 \in \operatorname{dom}(A)$. Due to Theorem 6.C.5(b) we thus have

$$0 = \mathfrak{a}(v,v) = \mathfrak{a}(v_1,v_1) + \mathfrak{a}(v_2,v_2) + \mathrm{i}\mathfrak{a}(v_1,v_2) - \mathrm{i}\mathfrak{a}(v_2,v_1).$$

By taking real parts and using that $\inf \mathrm{W}(\mathfrak{a}) = 0$, one thus obtains $\mathfrak{a}(v_1,v_1) = \mathfrak{a}(v_2,v_2) = 0$. Since v is non-zero, so is at least one of the vectors v_1, v_2. Thus by Theorem 6.C.5(b), we have found a real eigenvector of A for the eigenvalue 0. Let us call this eigenvector v from now on. Again Theorem 6.C.5(b) gives

$$0 = \mathfrak{a}(v,v) = \mathfrak{a}(v^+,v^+) + \mathfrak{a}(v^-,v^-) - 2\mathfrak{a}(v^-,v^+) \geq \mathfrak{a}(v^+,v^+) + \mathfrak{a}(v^-,v^-),$$

where we use that $\mathfrak{a}$ is real to get $\operatorname{Re}\mathfrak{a}(v^-,v^+) = \mathfrak{a}(v^-,v^+)$. Hence $\mathfrak{a}(v^+,v^+) = \mathfrak{a}(v^-,v^-) = 0$ since $\inf \mathrm{W}(\mathfrak{a}) = 0$. At least one of the vectors v^+, v^- is non-zero and is thus in $\ker A$ by Theorem 6.C.5(b). ☐

In contrast to Theorem 6.3.3, observe that $\mathrm{s}(A)$ in Theorem 6.C.6 need not be an isolated spectral value.

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[12] And hence for all $\lambda > \mathrm{s}(A)$ by Theorem 5.4.1.

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