

Chapter 5

Positive and eventually positive solutions to PDEs

5.1 Positivity via sesquilinear forms

In Example 3.3.6(b), we described the domain and the action of the Dirichlet Laplacian Δ_{Dir} in terms of the expression $\int_{\Omega} \overline{\nabla v} \cdot \nabla u \, dx$, defined for elements $u, v \in H_0^1(\Omega)$. This is a special case of a *sesquilinear form*. It is the purpose of this section to develop some key aspects in the theory of sesquilinear forms, which yield useful tools to study linear operators on Hilbert spaces. In addition, these so-called *form methods* are well-suited to the study of positivity, as shown in Theorem 5.1.7 below.

Definition 5.1.1 (Sesquilinear forms). Let V be a complex Hilbert space.

- (a) A map $\alpha: V \times V \rightarrow \mathbb{C}$ is called a **sesquilinear form** if it is antilinear in the first component¹ and linear in the second.
- (b) A sesquilinear form $\alpha: V \times V \rightarrow \mathbb{C}$ is called **bounded** if there exists a number $c \geq 0$ such that $|\alpha(v, w)| \leq c \|v\|_V \|w\|_V$ for all $v, w \in V$.

A fundamental result is that bounded sesquilinear forms can be represented by bounded linear operators.

Lemma 5.1.2 (Lax-Milgram). Let V be a complex Hilbert space and let $\alpha: V \times V \rightarrow \mathbb{C}$ be a bounded sesquilinear form.

- (a) There exists a unique $A \in \mathcal{L}(V)$ that satisfies $\alpha(w, v) = -(w | Av)_V$ for all $v, w \in V$.²
- (b) Moreover, if $\text{Re } \alpha(v, v) \geq \delta \|v\|_V^2$ for a number $\delta > 0$ and all $v \in V$, then A is bijective.

¹This is consistent with our convention for inner products.

²We put a minus sign here to be consistent with Definition 5.1.3 below.

Proof. (a) The uniqueness is clear. For existence, note that for each $v \in V$, $-\alpha(\cdot, v)$ is a bounded antilinear functional on V . By the Riesz–Fréchet theorem, there exists $Av \in V$ such that $-\alpha(w, v) = (w | Av)_V$ for all $v, w \in V$. Clearly, $v \mapsto Av$ is linear and

$$|(w | Av)_V| = |\alpha(w, v)| \leq c \|w\|_V \|v\|_V \quad \text{for all } w \in V$$

for some constant $c \geq 0$. Hence, $\|Av\|_V \leq c \|v\|_V$ for all $v \in V$, so A is bounded.

(b) We may assume that $V \neq \{0\}$. So, boundedness of A ensures that $\sigma(A)$ is non-empty and compact. Let $\lambda \in \sigma(A)$ with minimal real part. It suffices to show that $\operatorname{Re} \lambda > 0$, since this implies $0 \notin \sigma(A)$. As λ lies on the boundary of $\sigma(A)$, there exists an **approximate eigenvector** of A for λ , i.e. a normalised sequence (v_n) in V such that $\lambda v_n - Av_n \rightarrow 0$ (see Exercise 5.2). It follows that $\delta \leq \operatorname{Re} \lambda$, since

$$0 \leftarrow \operatorname{Re}(\lambda v_n - Av_n | v_n)_V = \operatorname{Re} \lambda - \operatorname{Re} \alpha(v_n, v_n) \leq \operatorname{Re} \lambda - \delta. \quad \square$$

For the Dirichlet Laplacian, given $u, f \in L^2(\Omega)$, we have $u \in \operatorname{dom}(\Delta_{\text{Dir}})$ and $\Delta_{\text{Dir}} u = f$ if and only if $u \in H_0^1(\Omega)$ and $\alpha(v, u) = -(v | f)_{L^2(\Omega)}$ for all $v \in H_0^1(\Omega)$ (Example 3.3.6(b)). This observation serves as a blueprint for a general way to obtain operators from forms.

Definition 5.1.3 (Operators induced by forms). Let V, H be complex Hilbert spaces such that V embeds continuously and densely into H . Let $\alpha: V \times V \rightarrow \mathbb{C}$ be a bounded sesquilinear form on V . The **operator associated to α** , $A: H \supseteq \operatorname{dom}(A) \rightarrow H$, is given by

$$\begin{aligned} \operatorname{dom}(A) &:= \{u \in V : \exists f \in H, \forall v \in V : \alpha(v, u) = -(v | f)_H\}, \\ Au &:= f, \end{aligned}$$

where $f \in H$ in the definition of Au is the vector that occurs in the definition of $\operatorname{dom}(A)$.³

There are two differences between Lemma 5.1.2 and Definition 5.1.3. In the lemma the inner product used to defined A is $(\cdot | \cdot)_V$ while it is $(\cdot | \cdot)_H$ in the definition. Consequently, the operator A in the lemma maps from V to V , while it maps from the smaller space $\operatorname{dom}(A)$ to the larger space H in the definition. We now show that the operator A from Definition 5.1.3 is quite well-behaved if α satisfies a so-called ellipticity estimate.

Theorem 5.1.4 (Properties of operators induced by forms). *Let V, H be complex Hilbert spaces such that V embeds continuously and densely into H . Let $\alpha: V \times V \rightarrow \mathbb{C}$ be a bounded sesquilinear form on V . Assume that α satisfies the **ellipticity estimate***

$$\operatorname{Re} \alpha(v, v) + \mu \|v\|_H^2 \geq \delta \|v\|_V^2 \quad (5.1.1)$$

for some numbers $\mu \in \mathbb{R}$ and $\delta > 0$ and for all $v \in V$. Then the operator $A: H \supseteq \operatorname{dom}(A) \rightarrow H$ associated to α has the following properties:

(a) A is closed and densely defined.

³Observe that f is uniquely determined since V is dense in H . Thus, A is indeed well-defined.

(b) One has $s(A) \leq \mu$ and every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \mu$ satisfies $\|\mathcal{R}(\lambda, A)\|_{H \rightarrow H} \leq \frac{1}{\operatorname{Re} \lambda - \mu}$.

(c) If the form \mathfrak{a} is **symmetric**, i.e. $\mathfrak{a}(u, v) = \overline{\mathfrak{a}(v, u)}$ for all $u, v \in V$, then $\sigma(A) \subseteq \mathbb{R}$.⁴

Proof. (a) Let $(u_n) \subseteq \operatorname{dom}(A)$ converge in H to $u \in H$ and assume (Au_n) converges to $v \in H$. Since $\mathfrak{a}(u_n - u_m, u_n - u_m) = (u_n - u_m | Au_n - Au_m)_H$, it follows from the ellipticity estimate (5.1.1) that (u_n) is Cauchy in V , hence convergent in V . The embedding $V \hookrightarrow H$ yields $u \in V$. The boundedness of \mathfrak{a} and the convergence $u_n \rightarrow u$ in V imply $\mathfrak{a}(w, u) = \lim_{n \rightarrow \infty} \mathfrak{a}(w, u_n) = -(w | v)_H$ for every $w \in V$. Hence, $u \in \operatorname{dom}(A)$ and $Au = v$, so A is indeed closed.

We now show (b). The density of $\operatorname{dom}(A)$ then follows from Exercise 5.1.

(b) Let $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \mu$. We first show that $\lambda - A: \operatorname{dom}(A) \rightarrow H$ is surjective, so let $f \in H$. Since $(\cdot | f)_H$ defines a bounded antilinear form on V , the Riesz–Fréchet representation theorem gives a vector $v_0 \in V$ such that $(u | v_0)_V = (u | f)_H$ for all $u \in H$. Now consider the bounded sesquilinear form $\mathfrak{b}: V \times V \rightarrow \mathbb{C}$, $\mathfrak{b}(u, v) := \lambda(u | v)_H + \mathfrak{a}(u, v)$ satisfies $\operatorname{Re} \mathfrak{b}(v, v) \geq \delta \|v\|_V^2$ for all $v \in V$ and let $B \in \mathcal{L}(V)$ be the associated bijective operator (Lemma 5.1.2(b)). Then the vector $w_0 := -B^{-1}v_0 \in V$ satisfies

$$\mathfrak{a}(u, w_0) + \lambda(u | w_0)_H = \mathfrak{b}(u, w_0) = (u | v_0)_V = (u | f)_H$$

for all $u \in V$. Hence, $w_0 \in \operatorname{dom}(A)$ and $(\lambda - A)w_0 = f$, so $\lambda - A$ is indeed surjective.

On the other hand, $\lambda - A$ is injective, hence bijective, because

$$\begin{aligned} \|v\|_H \|(\lambda - A)v\|_H &\geq \operatorname{Re}(v | \lambda v - Av)_H \\ &\geq \operatorname{Re} \lambda \|v\|_H^2 + \operatorname{Re} \mathfrak{a}(v, v) \geq (\operatorname{Re} \lambda - \mu) \|v\|_H^2 + \delta \|v\|_V^2. \end{aligned}$$

In fact, this even gives that $\|(\lambda - A)v\|_H \geq (\operatorname{Re} \lambda - \mu) \|v\|_H$ for all $v \in \operatorname{dom}(A)$, which implies the desired resolvent estimate if $\operatorname{Re} \lambda > \mu$.

(c) Consider the form $\mathfrak{b}: V \times V \rightarrow \mathbb{C}$ given by $\mathfrak{b}(u, v) := \mathfrak{a}(u, v) + \mu(u | v)_H$. The operator B associated to \mathfrak{b} can easily be checked to satisfy $B = A - \mu$ (with the same domain as A). So it suffices to show that $\sigma(B) \subseteq \mathbb{R}$.

The symmetry of \mathfrak{a} implies that $\mathfrak{a}(v, v) \in \mathbb{R}$ and thus $\mathfrak{b}(v, v) \geq \delta \|v\|_V^2$ for all $v \in V$. Now take a complex number γ with $\operatorname{Re} \gamma > 0$. Then the form $\gamma \mathfrak{b}: V \times V \rightarrow \mathbb{C}$, which is associated to the operator γB , satisfies

$$\operatorname{Re}(\gamma \mathfrak{b}(v, v)) = (\operatorname{Re} \gamma) \mathfrak{b}(v, v) \geq \operatorname{Re} \gamma \delta \|v\|_V^2$$

for all $v \in V$; here we used again that $\mathfrak{b}(v, v) \in \mathbb{R}$. We thus conclude from (b) that $s(\gamma B) \leq 0$, so $\operatorname{Re}(\gamma \lambda) \leq 0$ for every $\lambda \in \sigma(B)$. As this is true for each $\gamma \in \mathbb{C}$ with $\operatorname{Re} \gamma > 0$, it follows that $\sigma(B) \subseteq \mathbb{R}$. \square

⁴In fact, if \mathfrak{a} is symmetric, then one can show that A is a so-called **self-adjoint** operator, but we shall not discuss this further at this point.

We frequently use Theorem 5.1.4 to study examples, starting with Exercises 5.4 and 5.6 and Example 5.4.3. In Definition 4.2.9 we introduced real operators on complex Banach lattices. Analogously we now define real sesquilinear forms.

Definition 5.1.5. Let V be a Hilbert space that embeds continuously and densely into $L^2(\Omega, \nu)$ for a σ -finite measure-space (Ω, ν) . A sesquilinear form $\mathfrak{a}: V \times V \rightarrow \mathbb{C}$ is called **real** if $V_{\mathbb{R}} := V \cap L^2(\Omega, \nu; \mathbb{R})$ satisfies $V = V_{\mathbb{R}} + iV_{\mathbb{R}}$ and $\mathfrak{a}(u, v) \in \mathbb{R}$ for all $u, v \in V_{\mathbb{R}}$.

It is easy to see that the condition $V = V_{\mathbb{R}} + iV_{\mathbb{R}}$ in the previous definition is equivalent to the condition $\operatorname{Re} v \in V$ for all $v \in V$. The following proposition is a straightforward consequence of Definitions 4.2.9, 5.1.3, and 5.1.5.

Proposition 5.1.6 (Real forms induce real operators). *In the situation of Definition 5.1.5, if \mathfrak{a} is bounded and real, then so is the associated operator $A: L^2(\Omega, \nu) \supseteq \operatorname{dom}(A) \rightarrow L^2(\Omega, \nu)$.*

For operators constructed via real forms, the following result makes it often quite easy to check whether the resolvent is positive for all sufficiently large real numbers. The result is a substantial generalisation of Exercise 4.3. The inequality $\mathfrak{a}(v^-, v^+) \leq 0$ in assertion (ii) is an infinite-dimensional version of property (vii) in Exercise 1.2.

Theorem 5.1.7 (The Beurling–Deny criterion for positivity of resolvents). *Under the assumptions of Theorem 5.1.4, let $H = L^2(\Omega, \nu)$ for a σ -finite measure space (Ω, ν) . Assume in addition that \mathfrak{a} is real. Then the following are equivalent for the associated operator $A: L^2(\Omega, \nu) \supseteq \operatorname{dom}(A) \rightarrow L^2(\Omega, \nu)$:*

- (i) $\mathcal{R}(\lambda, A) \geq 0$ for all $\lambda \in [\mu, \infty)$.
- (ii) $V_{\mathbb{R}} := V \cap L^2(\Omega, \nu; \mathbb{R})$ is a sublattice of $L^2(\Omega, \nu; \mathbb{R})$ and $\mathfrak{a}(v^-, v^+) \leq 0$ for all $v \in V_{\mathbb{R}}$.

Proof. Observe that, as \mathfrak{a} is real by assumption, so is A according to Proposition 5.1.6. This easily implies that $\mathcal{R}(\lambda, A)$ is also a real operator for every $\lambda \in \mathbb{R} \cap \rho(A)$.

“(ii) \Rightarrow (i)”: Let $\lambda \in [\mu, \infty)$, $0 \leq f \in L^2(\Omega)$, and set $v := \mathcal{R}(\lambda, A)f$. We know already that $v \in V_{\mathbb{R}}$, and need to show that $v \geq 0$. To this end, we now proceed similarly as in Exercise 4.3. Since $v^- \in V$, one can compute using $(v^- | v^+)_{\mathcal{H}} = 0$ that

$$\begin{aligned} 0 \leq (v^- | f)_{\mathcal{H}} &= (v^- | (\lambda - A)v)_{\mathcal{H}} \\ &= -\lambda \|v^-\|_{\mathcal{H}}^2 + \mathfrak{a}(v^-, v) \leq -\mu \|v^-\|_{\mathcal{H}}^2 - \mathfrak{a}(v^-, v^-) \leq -\delta \|v^-\|_{\mathcal{V}}^2; \end{aligned}$$

where the inequalities respectively used $\mathfrak{a}(v^-, v^+) \leq 0$ and ellipticity estimate (5.1.1). Hence $v^- = 0$, which yields $v \geq 0$.

“(i) \Rightarrow (ii)”: Let $v \in V_{\mathbb{R}}$. To show that $V_{\mathbb{R}}$ is a sublattice, it suffices to prove that $v^+ \in V$. For this we use the following abstract regularisation technique: set $w_n := n\mathcal{R}(n, A)v^+$ for all $n > \mu^+$. Then $(w_n) \subseteq \operatorname{dom}(A) \subseteq V$ and, as shown in Exercise 5.1, $w_n \rightarrow v^+$ in L^2 -norm. In particular, (w_n) is bounded in H . Assume for a moment that (w_n) is also bounded in V . As V is reflexive, a subsequence (w_{n_k}) converges weakly to some $w \in V$ in V , hence in H . The uniqueness of weak limits in H ensures $v^+ = w \in V$.

Let us thus prove that the norms $\|w_n\|_V$ are uniformly bounded. Observe that

$$\begin{aligned} \mathfrak{a}(v - w_n, w_n) &= -(v - w_n | Aw_n)_H = n(v - w_n | v^+ - w_n)_H \\ &= n\left(\|v^+ - w_n\|_H^2 + (v^- | w_n)_H\right) \geq 0 \end{aligned}$$

since $w_n \geq 0$ due to (i). Thus, $\mathfrak{a}(w_n, w_n) \leq \mathfrak{a}(v, w_n)$ for all n . Using the ellipticity estimate (5.1.1), this implies

$$\begin{aligned} \delta \|w_n\|_V^2 &\leq \mathfrak{a}(w_n, w_n) + \mu \|w_n\|_H^2 \\ &\leq \mathfrak{a}(v, w_n) + \mu^+ \|w_n\|_H^2 \leq c \|v\|_V \|w_n\|_V + \mu^+ d \|w_n\|_H \|w_n\|_V \end{aligned}$$

for constants $c, d \geq 0$ and all indices $n > \mu^+$; for the last inequality, we used that \mathfrak{a} is bounded and $V \hookrightarrow H$. Dividing by $\|w_n\|_V$ yields that (w_n) is bounded in V .

It remains to show the second claim in (ii), i.e. that $\mathfrak{a}(v^-, v^+) \leq 0$. Note that

$$\mathfrak{a}(v^-, w_{n_k}) = -(v^- | Aw_{n_k})_H = -n(v^- | w_{n_k} - v^+)_H = -n(v^- | w_{n_k}) \leq 0$$

for each n . Since $\mathfrak{a}(v^-, \cdot)$ is a bounded linear functional on V , the weak convergence of (w_{n_k}) to v^+ in V implies that $0 \geq \mathfrak{a}(v^-, w_{n_k}) \rightarrow \mathfrak{a}(v^-, v^+)$. \square

As a first application of Theorem 5.1.7, we will discuss Laplace operators with non-local boundary conditions in Exercise 5.6 and Example 5.4.3.

5.2 The maximum principle

In this and the next section, we present a different way to obtain positivity of solution operators to certain PDEs, based on the **maximum principle**. As a motivation, consider a function $v \in C^2([0, 1]; \mathbb{R})$ that satisfies $v'' \geq 0$. This means that v is convex, and hence its maximal value is attained at least at one of the boundary points $\{0, 1\}$.

The maximum principle generalises this to a larger class of operators, also on higher dimensional domains. There, one cannot directly work with convexity of v , but the Laplace operator also has a related property that translates well into more general situations: if $0 \leq v \in C^2([0, 1]; \mathbb{R})$ vanishes at a point $x \in (0, 1)$, then $\Delta v(x) \geq 0$. This property is captured by assumption (1) in the following theorem. For a first intuition, one should think in Theorem 5.2.1 of the situation where $S := \Omega$ is an open set in \mathbb{R}^n and $M = \overline{\Omega}$. Non-open S will become relevant in Chapter 6.

Theorem 5.2.1 (An abstract maximum principle). *Let (M, d) be a metric space and let $\emptyset \neq S \subseteq M$ be relatively compact.⁵ Let $D \subseteq C(\overline{S}; \mathbb{R})$ be a vector subspace such that $\mathbb{1} := \mathbb{1}_{\overline{S}} \in D$ and let $A: D \rightarrow \mathbb{R}^S$ be a linear map with the following properties:*

- (1) *The map A satisfies the **positive minimum principle on S** , i.e. for each $x \in S$ and each function $0 \leq u \in D$ one has the implication*

$$u(x) = 0 \quad \implies \quad (Au)(x) \geq 0.$$

⁵Recall that a subset of a metric space is called **relatively compact** if its closure is compact.

(2) One has $A\mathbb{1} \leq 0$ and there exists a function $0 \leq w \in D$ with $(Aw)(x) > 0$ for all $x \in S$.

Let $v \in D$ attain at least one value in $[0, \infty)$ and satisfy $Av \geq 0$. Then $\partial S \neq \emptyset$ and v attains its maximum at ∂S .

Before the proof, we show Theorem 5.2.1 in action for a classic PDE example.

Example 5.2.2 (The weak maximum principle for the Laplace operator). Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open and bounded and let $c \in (-\infty, 0]$. Assume that $v \in C(\overline{\Omega}; \mathbb{R}) \cap C^2(\Omega; \mathbb{R})$ satisfies $\Delta v(x) + cv(x) \geq 0$ for all $x \in \Omega$.

If v has at least one value in $[0, \infty)$ then v attains its maximum on $\partial\Omega$. In particular, if v vanishes on $\partial\Omega$, then $v \leq 0$ in Ω .

Proof. We apply Theorem 5.2.1 to the set $S := \Omega$ in the metric space $M := \overline{\Omega}$. Let $D := C(\overline{\Omega}; \mathbb{R}) \cap C^2(\Omega; \mathbb{R}) \subseteq C(\overline{\Omega}; \mathbb{R})$ and define $A: D \rightarrow \mathbb{R}^\Omega$ by $Au := (\Delta + c)u|_\Omega$ for all $u \in D$. It suffices to show that A satisfies the assumptions (1) and (2) of Theorem 5.2.1.

(1) Let $x \in \Omega$ and let $0 \leq u \in D$ satisfy $u(x) = 0$. Then u has a global minimum at x . As u is C^2 in a neighbourhood of x , it follows that the Hessian matrix $Hu(x)$ is positive semidefinite. So its trace satisfies $\text{tr}(Hu(x)) \geq 0$ and thus,

$$(Au)(x) = \Delta u(x) + cu(x) = \text{tr}(Hu(x)) \geq 0.$$

(2) Clearly, $A\mathbb{1} = c\mathbb{1} \leq 0$. Let $w \in D$ be given by $w(x) = e^{\alpha x_1}$ for all $x \in \overline{\Omega}$ and a real number α that satisfies $\alpha^2 > -c$. Then $(Aw)(x) = (\alpha^2 + c)w(x) > 0$ for all $x \in \Omega$. \square

We will discuss below in Example 5.3.6 how the weak maximum principle is related to the positivity of $\mathcal{R}(\lambda, \Delta_{\text{Dir}})$. Theorem 5.2.1 can be applied to further types of differential operators, see Exercise 5.3. For the proof of the theorem, we use the following lemma.

Lemma 5.2.3. *Let (M, d) be a compact metric space and let $\emptyset \neq K \subseteq M$ be a compact subset. Let (f_k) be a sequence in $C(M; \mathbb{R})$ that converges uniformly to a function $f \in C(M; \mathbb{R})$. If each f_k attains its maximum on K , then so does f .*

Proof. By assumption, for each $k \in \mathbb{N}$ there is a point $x_k \in K$ such that f_k attains its maximum at x_k . As K is compact, by passing to a subsequence we may assume that (x_k) converges to a point $x^* \in K$. For every $x \in M$, by continuity of f one then has

$$f(x) \leq f_k(x) + \|f - f_k\|_\infty \leq f_k(x_k) + \|f - f_k\|_\infty \leq f(x_k) + 2\|f - f_k\|_\infty \xrightarrow{k \rightarrow \infty} f(x^*). \quad \square$$

Proof of Theorem 5.2.1. We first make a preliminary observation: If a function $u \in D$ attains its maximum at a point $x_1 \in S$ and satisfies $u(x_1) \geq 0$, then $(Au)(x_1) \leq 0$.

Indeed, set $h := u(x_1)\mathbb{1} - u$. Then $0 \leq h \in D$ and $h(x_1) = 0$. According to assumption (1), A satisfies the positive minimum principle in S , so it follows that $(Ah)(x_1) \geq 0$ and thus $(Au)(x_1) \leq u(x_1) \cdot (A\mathbb{1})(x_1)$. As $u(x_1) \geq 0$ and since $(A\mathbb{1})(x_1) \leq 0$ by assumption (2), it follows that $(Au)(x_1) \leq 0$, as claimed.

Now, define $v_k := v + \frac{1}{k}w \in D$ for each $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ choose a point $x_k \in \bar{S}$ where v_k attains its maximum. Fix an index $k \in \mathbb{N}$ for the moment. Since $w \geq 0$ and by assumption v is not everywhere < 0 in \bar{S} , one has $v_k(x_k) \geq 0$. Moreover, $Av \geq 0$ implies $(Av_k)(x) \geq \frac{1}{k}(Aw)(x) > 0$ for all $x \in S$, so it follows from the preliminary observation that $x_k \notin S$, so in particular $x_k \in \partial S$. So each v_k attains its maximum at ∂S and Lemma 5.2.3 shows that the same is true for v . \square

5.3 Intermezzo: Regularity of solutions

In this section we briefly discuss – without proofs, but with a few intuitive explanations – that the solution u of the equation $(\lambda - \Delta_{\text{Dir}})u = f$ has better regularity than f . We need this later in the course; for the moment we focus on the following motivation.

We know from Exercise 4.3 that the resolvent $\mathcal{R}(\lambda, \Delta_{\text{Dir}}): L^2(\Omega) \rightarrow L^2(\Omega)$ is positive if $\lambda > 0$, and the technique from this exercise was generalised to a very broad class of operators in the Beurling–Deny criterion in Theorem 5.1.7. It is natural to ask whether one can alternatively use the maximum principle from Section 5.2 to obtain positivity of $\mathcal{R}(\lambda, \Delta_{\text{Dir}})$. There is a difficulty with this approach: for $f \in L^2(\Omega)$, the function $u = \mathcal{R}(\lambda, \Delta_{\text{Dir}})f$ will in general only be in $\text{dom}(\Delta_{\text{Dir}})$, but the maximum principle (Example 5.2.2) requires more from u , namely to be continuous on $\bar{\Omega}$ and C^2 on Ω .

The question thus arises: under what conditions is the solution u sufficiently smooth? This is a fundamental question of the **regularity theory** for PDEs, which is a rather subtle subject: not only does the answer depend on f , but also on the geometric properties of Ω . For this reason, it is essential to be able to quantify ‘smoothness’ of the boundary.

Definition 5.3.1. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be bounded and open. For $k \in \mathbb{N}$, we say that Ω has **C^k boundary**, or equivalently, that $\partial\Omega$ is of **class C^k** , if there exists $\Phi \in C^k(\mathbb{R}^n; \mathbb{R})$ such that⁶

$$\Omega = \{x \in \mathbb{R}^n : \Phi(x) > 0\}$$

and $\nabla\Phi(x) \neq 0$ for all $x \in \partial\Omega$. It follows that $\partial\Omega$ is the level set $[\Phi = 0]$.

We can now state the major regularity result for the Dirichlet Laplacian Δ_{Dir} on $L^2(\Omega)$.

Theorem 5.3.2 (Elliptic regularity for the Dirichlet Laplacian). *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open with a bounded C^2 boundary. Let $\lambda > 0$. Then the resolvent $\mathcal{R}(\lambda, \Delta_{\text{Dir}})$ of the Dirichlet Laplacian on $L^2(\Omega)$ has the following properties:*

- (a) $\text{dom}(\Delta_{\text{Dir}}) = \mathcal{R}(\lambda, \Delta_{\text{Dir}})L^2(\Omega) \subseteq H^2(\Omega)$.
- (b) More generally, for every integer $k \geq 0$ one has $\mathcal{R}(\lambda, \Delta_{\text{Dir}})H^k(\Omega) \subseteq H^{k+2}(\Omega)$.

This is a non-trivial result that is best left for a dedicated course in PDEs. Thus we do not prove it here and refer instead to the literature, e.g. [Bre11, Theorem 9.25]. However, it is instructive to discuss a few special cases of assertion (a):

⁶Some readers may be familiar with the definition of C^k boundary via local charts. We comment on this further in the notes at the end of the chapter.

Remarks 5.3.3. In the setting of Theorem 5.3.2, let $u \in \text{dom}(\Delta_{\text{Dir}})$. Assertion (a) of the theorem says that $u \in H^2(\Omega)$. Let us explain this in the following simpler situations:

- (a) If $n = 1$, the fact that $u \in H^2(\Omega)$ is not surprising at all: by the definition of $\text{dom}(\Delta_{\text{Dir}})$, one has $u \in H_0^1(\Omega)$ and $\Delta u \in L^2(\Omega)$, so the first and second weak derivatives of u exist and are in $L^2(\Omega)$. Thus $u \in H^2(\Omega)$. The point here is that, since Ω is one-dimensional, there is only one second derivative and it equals Δu .
- (b) Now let $n \geq 2$ but consider $\Omega = \mathbb{R}^n$. Again, we know $u \in H_0^1(\mathbb{R}^n)$ and $\Delta u \in L^2(\mathbb{R}^n)$. But why does $\Delta u \in L^2(\mathbb{R}^n)$ imply that the weak derivatives $\partial_j \partial_k u$ exist⁷ and are in $L^2(\mathbb{R}^n)$ for all j, k ? The key insight is as follows:

Assume for a moment that also $u \in C_c^\infty(\mathbb{R}^n)$. One readily checks that Δ_{Dir} acts as the classical Laplace operator on u . Moreover, integration by parts with respect to the j th and the k th variables shows that $\int_{\mathbb{R}^n} (\partial_j^2 \bar{u})(\partial_k^2 u) \, dx = \int_{\mathbb{R}^n} |\partial_j \partial_k u|^2 \, dx$ for all indices j, k . Hence, $\|\Delta_{\text{Dir}} u\|_{L^2}^2 = \sum_{j,k=1}^n \|\partial_j \partial_k u\|_{L^2}^2$ and thus,

$$\|\Delta_{\text{Dir}} u\|_{L^2}^2 + \|u\|_{H^1}^2 = \|u\|_{H^2}^2 \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n). \quad (5.3.1)$$

One can prove – although we shall not do this here – that $C_c^\infty(\mathbb{R}^n)$ is in dense $\text{dom}(\Delta_{\text{Dir}})$ with respect to $\|\cdot\|_{\Delta_{\text{Dir}}}$, so (5.3.1) implies that $\text{dom}(\Delta_{\text{Dir}}) \subseteq H^2(\mathbb{R}^n)$.

- (c) On domains $\Omega \subseteq \mathbb{R}^n$, things are more involved. Clearly, (5.3.1) holds for $u \in C_c^\infty(\Omega)$, but this space will typically not be dense in $\text{dom}(\Delta_{\text{Dir}})$. Indeed, if u is the closure of $C_c^\infty(\Omega)$ within $\text{dom}(\Delta_{\text{Dir}})$, then (5.3.1) implies $u \in H_0^2(\Omega)$, which is a proper subspace of $\text{dom}(\Delta_{\text{Dir}})$ in general.

Theorem 5.3.2 is particularly useful when combined with the following result that connects weak to classical differentiability.

Theorem 5.3.4 (A Sobolev embedding theorem). *Let $n \geq 2$ and let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open and bounded with C^1 boundary. Let $k \in \mathbb{N}$ and $p \in [1, \infty]$ satisfy $n < kp$ and let $m \in \mathbb{N}_0$ satisfy $m < k - \frac{n}{p}$. Then $W^{k,p}(\Omega) \hookrightarrow C^m(\bar{\Omega})$.*

For $n = 1$ a bit more is true, see Theorem 5.3.7(b) below. A proof of Theorem 5.3.4 can be found, for example, in [Eva10, Section 5.6.3]. However, the result is true under weaker regularity assumptions on Ω ; we present some details for the interested reader in Theorem 5.B.8 of the supplementary Section 5.B.

Theorems 5.3.2 and 5.3.4 can be used to derive positivity of $\mathcal{R}(\lambda, \Delta_{\text{Dir}})$, in some cases, from the maximum principle (as a classical alternative to the form approach). To demonstrate this, we use the following properties.

Proposition 5.3.5. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open and $1 \leq p < \infty$.*

- (a) *The cone of positive test functions, $C_c^\infty(\Omega) \cap L^p(\Omega)_+$, is dense in $L^p(\Omega)_+$.*

⁷We use here the common notation $\partial_j \partial_k u$ for the weak partial derivative with respect to the k th and the j th variable. In terms of the multi-index notation used in Definition 3.2.4 this means $\partial_j \partial_k u := \partial^{e_j + e_k} u$.

- (b) Let $n \geq 2$ and assume that Ω is bounded and has C^1 boundary. Then every $u \in C(\bar{\Omega}) \cap W_0^{1,p}(\Omega)$ vanishes on $\partial\Omega$.

Again, we refrain from showing the proofs. Assertion (a) can be obtained by means the standard cut-off and regularisation technique which is, for instance, also used in Supplement 3.A. Assertion (b) reflects the intuition discussed after Example 3.3.6: functions in $H_0^1(\Omega)$ “vanish” on $\partial\Omega$. The proof is based on the so-called trace operator that is explained in a bit more detail in Supplement 3.B, where we also discuss Ω with less regular boundary.

The conclusion of the next example is already known from Exercise 4.3, even without regularity or boundedness assumptions on Ω . Yet, the example seems worthwhile as it demonstrates the connection between the maximum principle and positive resolvents.

Example 5.3.6 (Positivity of the resolvent of the Dirichlet Laplacian, again). Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^2 boundary. Then for every $\lambda > 0$ the resolvent of the Dirichlet Laplacian on $L^2(\Omega)$ satisfies $\mathcal{R}(\lambda, \Delta_{\text{Dir}}) \geq 0$.

Proof. Fix $\lambda > 0$. It follows from Theorem 5.3.2(b) that $\mathcal{R}(\lambda, \Omega) C_c^\infty(\Omega) \subseteq H^\ell(\Omega)$ for every $\ell \in \mathbb{N}_0$. Now choose an integer $k \in \mathbb{N}$ such that $2 < k - \frac{n}{2}$. Then

$$\mathcal{R}(\lambda, \Omega) C_c^\infty(\Omega) \subseteq H^k(\Omega) \subseteq C^2(\bar{\Omega}),$$

where the second inclusion is due to the Sobolev embedding theorem 5.3.4. By Proposition 5.3.5(a), it suffices to show that $\mathcal{R}(\lambda, \Omega)$ maps $C_c^\infty(\Omega) \cap L^2(\Omega)_+$ into $L^2(\Omega)_+$.

Let $0 \leq f \in C_c^\infty(\Omega)$ and set $u := \mathcal{R}(\lambda, \Delta_{\text{Dir}}) f$. Then $u \in C^2(\bar{\Omega})$ as shown above. In particular, u is continuous on $\bar{\Omega}$. Since $u \in \text{dom}(\Delta_{\text{Dir}}) \subseteq H_0^1(\Omega)$, we conclude that u vanishes on $\partial\Omega$ (Proposition 5.3.5(b)). Since $(\Delta_{\text{Dir}} - \lambda)(-u) = f \geq 0$, the weak maximum principle for the Laplace operator (Example 5.2.2) implies that $-u \leq 0$, so $u \geq 0$. \square

The observant reader might have noticed that the regularity improvement in Theorem 5.3.2(b) is more than we needed in Example 5.3.6: there, it suffices to know that the regularity of $\mathcal{R}(\lambda, \Delta_{\text{Dir}}) f$ is not worse than that of f , i.e. if $\mathcal{R}(\lambda, \Delta_{\text{Dir}}) H^k(\Omega) \subseteq H^k(\Omega)$ for each $k \geq 0$. We will, however, use the stronger result of Theorem 5.3.2 in later examples.

We end this subsection with the following theorem on the regularity of Sobolev functions in dimension 1. It shows that, to a large extent, one can compute with weak derivatives in one dimension similarly as with classical derivatives.

Theorem 5.3.7. Let $\emptyset \neq I = (a, b) \subseteq \mathbb{R}$ be a bounded open interval and let $p \in [1, \infty]$.

- (a) Fundamental theorem of calculus: If $u \in W^{1,p}(I)$, then u has a representative \tilde{u} that is continuous on \bar{I} such that

$$\tilde{u}(x) = \tilde{u}(y) + \int_y^x u'(t) dt \quad \text{for } x, y \in \bar{I}. \quad (5.3.2)$$

- (b) Sobolev embedding: For all $k \in \mathbb{N}$, we have the continuous embedding $W^{k,p}(I) \hookrightarrow C^{k-1}(\bar{I})$. More precisely, $W^{k,p}(I) \hookrightarrow L^\infty(I)$ and every $u \in W^{k,p}(I)$ has a representative in $C^{k-1}(\bar{I})$.⁸
- (c) Integration by parts: If $u, v \in W^{1,p}(I)$, then $uv \in W^{1,p}(I)$ and

$$\int_y^x uv' \, dt = (u(x)v(x) - u(y)v(y)) - \int_y^x u'v \, dt, \quad \text{for } x, y \in [a, b].$$

Throughout the course, Theorem 5.3.7 will be used in a variety of examples. Hence, instead of a mere reference to the literature, we present the interested reader with proofs of parts (a) and (b) and a brief sketch of the proof of (c) in Supplement 5.A.

5.4 Positivity close to the spectral bound

For the Dirichlet Laplacian $\Delta_{\text{Dir}}: L^2(\Omega) \supseteq \text{dom}(\Delta_{\text{Dir}}) \rightarrow L^2(\Omega)$ on an open set $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ we know from Exercise 4.3 that $\mathcal{R}(\lambda, \Delta_{\text{Dir}}) \geq 0$ for all $\lambda > 0$. If Ω is bounded, one can even show that $s(\Delta_{\text{Dir}}) < 0$ ⁹, which raises the question of whether one also has positivity of the resolvent at all points $\lambda \in (s(\Delta_{\text{Dir}}), 0]$. This answer is affirmative, and one way to see this is the following general theorem (for $Q = 0$).

For elements x, y of a Banach lattice E we continue to use the convention introduced in Notation 1.2.4: we write $x \leq y$ if there exists a number $c > 0$ such that $x \leq cy$. For two operators $T, S \in \mathcal{L}(E, F)$ between Banach lattices, we write $S \leq T$ if $T - S \geq 0$ and, in the case of complex scalars, both operators are real. Naturally, we also write $S \leq T$ if there is a number $c > 0$ such that $S \leq cT$.

Theorem 5.4.1. *Let $A: E \supseteq \text{dom}(A) \rightarrow E$ be a closed linear operator on a complex Banach lattice E and let $0 \leq Q \in \mathcal{L}(E)$ be a projection. Let $\lambda_0, \mu_0 \in \mathbb{R}$ satisfy $\lambda_0 < \mu_0$ and $(\lambda_0, \mu_0] \subseteq \rho(A)$. If $\mathcal{R}(\mu_0, A) \geq Q$, then*

$$\mathcal{R}(\mu, A) \geq Q \quad \text{for all } \mu \in (\lambda_0, \mu_0].$$

Proof. Consider the set $U := \{\mu \in (\lambda_0, \mu_0) : \mathcal{R}(\lambda, A) \geq Q \text{ for all } \lambda \in (\mu, \mu_0)\}$. We want to show that $U = (\lambda_0, \mu_0)$. As the latter is connected, it suffices to show that U is non-empty, open, and closed within (λ_0, μ_0) . The closedness is straightforward to check. The non-emptiness follows from the Taylor expansion formula of $\mathcal{R}(\cdot, A)$ in Proposition 3.3.2(a): for all $\lambda < \mu_0$ sufficiently close to μ_0 one has

$$\mathcal{R}(\lambda, A) = \sum_{k=0}^{\infty} \underbrace{(\mu_0 - \lambda)^k}_{\geq 0} \underbrace{\mathcal{R}(\mu_0, A)^{k+1}}_{\geq Q^{k+1} = Q} \geq Q.$$

The same Taylor series argument can be used to show that if $\mu \in U$, then a left neighbourhood of μ is also in U . Since $[\mu, \mu_0) \subseteq U$ as well, U is indeed open. \square

⁸From now on we identify each $u \in W^{k,p}(I)$ with its representative in $C^{k-1}(\bar{I})$.

⁹This is a consequence of the **Poincaré inequality**; however, we leave this subject aside for now.

The fact that Q can be an arbitrary positive projection in Theorem 5.4.1 will become relevant later. For now, we note that even the case $Q = 0$ can be interesting:

Example 5.4.2 (Positivity of the resolvent for the Dirichlet Laplacian left of zero). Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open. The Dirichlet Laplacian $\Delta_{\text{Dir}}: L^2(\Omega) \ni \text{dom}(\Delta_{\text{Dir}}) \rightarrow L^2(\Omega)$ satisfies $\sigma(\Delta_{\text{Dir}}) \subseteq (-\infty, 0]$ and $\mathcal{R}(\lambda, \Delta_{\text{Dir}}) \geq 0$ for all $\lambda \in (s(\Delta_{\text{Dir}}), \infty)$.

Proof. The property $\sigma(\Delta_{\text{Dir}}) \subseteq (-\infty, 0]$ is proved in Exercise 5.4. Moreover, you showed in Exercise 4.3 that $\mathcal{R}(\lambda, A) \geq 0$ for all $\lambda \in (0, \infty)$. By applying Theorem 5.4.1 for $Q = 0$ and any $\mu_0 > 0$, one sees that this inequality remains true for all $\lambda > s(\Delta_{\text{Dir}})$. \square

Theorem 5.4.1 for $Q = 0$ shows that if the resolvent of an operator is positive at one point μ_0 , then it is also positive on the left of μ_0 up to the next spectral value on the real axis. On the other hand, the theorem gives no information on positivity on the right of μ_0 . The next example shows that there are even second-order differential operators A whose resolvent is positive in a right neighbourhood of $s(A)$, but not on all of $(s(A), \infty)$.

Example 5.4.3 (A Laplacian with non-local boundary conditions). Consider the sesquilinear form $\mathfrak{a}: H^1(0, 1) \times H^1(0, 1) \rightarrow \mathbb{C}$ defined by

$$\mathfrak{a}(v, u) := \int_0^1 \overline{v'} u' \, dx + \frac{1}{2} \begin{pmatrix} \overline{v(0)} & \overline{v(1)} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \end{pmatrix}. \quad (5.4.1)$$

Denote the associated operator by $\Delta_B: L^2(0, 1) \ni \text{dom}(\Delta_B) \rightarrow L^2(0, 1)$.

(a) The operator Δ_B is closed and acts as the weak second derivative on its domain

$$\text{dom}(\Delta_B) = \left\{ u \in H^2(0, 1) : \begin{pmatrix} -u'(0) \\ u'(1) \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \end{pmatrix} \right\}.$$

(b) One has $\sigma(\Delta_B) \subseteq (-\infty, 0)$.

(c) One has $\mathcal{R}(\lambda, \Delta_B) \geq 0$ for all $\lambda \in (s(\Delta_B), 0]$, but not for all $\lambda > 0$.

Proof. The form \mathfrak{a} is the one given in Exercise 5.6 for the choice $B := -\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (carefully note the minus sign in the form in that exercise).

(a) This is shown for general B in Exercise 5.6.

(b) As B is a self-adjoint matrix, the form \mathfrak{a} is symmetric in the sense of Theorem 5.1.4(c), and hence $\sigma(\Delta_B) \subseteq \mathbb{R}$. Moreover, as B is negative semidefinite, one can use any number $\mu > 0$ in the ellipticity estimate in Exercise 5.6(b). Hence, $s(\Delta_B) \leq 0$. It remains to show that $0 \notin \sigma(\Delta_B)$. To this end, we prove that $-\Delta_B: \text{dom}(\Delta_B) \rightarrow L^2(0, 1)$ is bijective.

If $u \in \ker(-\Delta_B)$, then $u \in H^2(0, 1)$ and $u'' = 0$. Thus, in fact $u \in H^k(0, 1)$ for all $k \in \mathbb{N}$, in particular $u \in C^2([0, 1])$ by Theorem 5.3.7(b). By using classical derivatives one can now immediately check that $u'' = 0$ and the boundary conditions in $\text{dom}(\Delta_B)$ imply $u = 0$. So $-\Delta_B$ is injective.

On the other hand, let $f \in L^2(0, 1)$. Define a function $u \in L^2(0, 1)$ by

$$u(x) = \frac{1}{4} \int_0^1 f(z) \, dz + \frac{1}{2} \int_x^1 \int_0^y f(z) \, dz \, dy + \frac{1}{2} \int_0^x \int_y^1 f(z) \, dz \, dy \quad (5.4.2)$$

for all $x \in [0, 1]$. By using the fundamental theorem of calculus for H^2 (see Theorem 5.3.7(a)) one can check that $u \in \text{dom}(\Delta_B)$ and $-\Delta_B u = f$. So $-\Delta_B$ is surjective.

- (c) Clearly, if f is positive, then so is u in formula (5.4.2), and hence $\mathcal{R}(0, \Delta_B)$ is a positive operator. Theorem 5.4.1 for $Q = 0$ thus implies that $\mathcal{R}(\mu, A) \geq 0$ for all $\mu \in (s(A), 0]$. On the other hand, the matrix B has a strictly negative off-diagonal entry, so Exercise 5.6(c) shows that $\mathcal{R}(\lambda, \Delta_B) \not\geq 0$ for some $\lambda > s(\Delta_B)$. \square

The boundary conditions in $\text{dom}(\Delta_B)$ in Example 5.4.3 are a simple example of **non-local Robin boundary conditions**. One can interpret them in more physical terms: the outward flux through the boundary points, represented by the vector $(-u'(0) \ u'(1))^T$, is equal to the average of the boundary values.

In Example 5.4.3 the resolvent $\mathcal{R}(\lambda, \Delta_B)$ is positive in a right neighbourhood of the spectral bound, but not for large $\lambda \in \mathbb{R}$. We have already encountered a similar behaviour in finite dimensions in Theorem 2.3.1(ii). In infinite dimensions we will study this phenomenon in more generality in the next chapter.

Exercises for Chapter 5

Exercise 5.1 (Approximation by resolvents). Let $A: X \ni \text{dom}(A) \rightarrow X$ be a closed operator on a complex Banach space X . Assume that there exist $\lambda_0 \in \mathbb{R}$, $C \geq 0$ such that

$$[\lambda_0, \infty) \subseteq \rho(A) \quad \text{and} \quad \|\lambda \mathcal{R}(\lambda, A)\| \leq C \quad \text{for all } \lambda \in [\lambda_0, \infty).$$

Prove the following statements:

- (a) $\lambda \mathcal{R}(\lambda, A)x \rightarrow x$ as $\lambda \rightarrow \infty$ for all $x \in \overline{\text{dom}(A)}$.
- (b) If X is reflexive¹⁰, then A is densely defined.

Hints: For $x \in X$, use the equality $n\mathcal{R}(n, A)x - x = A\mathcal{R}(n, A)x$ for all integers $n \geq \lambda_0$. Also observe that the graph of A is a closed convex subset of $X \times X$ and is hence weakly closed by the Hahn–Banach separation theorem.

Exercise 5.2 (Approximate eigenvectors for the boundary of the spectrum). Let $A: X \ni \text{dom}(A) \rightarrow X$ be a closed operator on a complex Banach space X . Fix $\lambda \in \partial\sigma(A)$ and let (λ_n) be a sequence in $\rho(A)$ that converges to λ . Show that there exists a sequence $(y_n) \subseteq X$ such that $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and $(\lambda - A)y_n \rightarrow 0$ in X as $n \rightarrow \infty$. Such a sequence is called an **approximate eigenvector** for the spectral value λ .

Hint: Use Proposition 3.3.2 to first obtain a sequence $(x_n) \subseteq X$ with $\|x_n\| = 1$ and $\|\mathcal{R}(\lambda_n, A)x_n\| \rightarrow \infty$ as $n \rightarrow \infty$.

Exercise 5.3 (The maximum principle for first order differential operators). Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty open set such that $0 \notin \Omega$, and let $b: \Omega \rightarrow \mathbb{R}^n$ be a vector field such that

$$x^\top b(x) > 0 \quad \text{for all } x \in \Omega. \tag{5.4.3}$$

For $D := C^1(\Omega; \mathbb{R})$, consider the linear operator $A: D \rightarrow \mathbb{R}^\Omega$ defined by

$$Af := b^\top \nabla f, \quad f \in D.$$

Show that if $u \in D$ satisfies $Au \geq 0$ in Ω , then u attains its maximum on $\partial\Omega$.

Hint: use (5.4.3) to design a function $0 \leq w \in D$ such that $Aw(x) > 0$ for all $x \in \Omega$.

¹⁰Recall that a Banach space X is reflexive if and only if every bounded sequence in X has a weakly convergent subsequence (due to the Eberlein–Šmulian theorem).

Exercise 5.4. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open. According to Example 3.3.6(c) no strictly positive number belongs to the spectrum of the Dirichlet Laplacian $\Delta_{\text{Dir}}: L^2(\Omega) \ni \text{dom}(\Delta_{\text{Dir}}) \rightarrow L^2(\Omega)$. Use Theorem 5.1.4 and the characterisation of Δ_{Dir} in Example 3.3.6(b) to show that even $\sigma(\Delta_{\text{Dir}}) \subseteq (-\infty, 0]$.

Exercise 5.5 (Estimates via maximum principle). Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be bounded and open. For $\lambda > 0$ and $f \in C(\overline{\Omega})_+$, assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solves the boundary value problem

$$\begin{cases} \lambda u - \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Prove that $0 \leq u(x) \leq \frac{1}{\lambda} \|f\|_\infty$ for all $x \in \overline{\Omega}$.

Exercise 5.6 (Laplacian with non-local boundary conditions). Let $B \in \mathbb{R}^{2 \times 2}$, and consider the sesquilinear form $\mathfrak{a}: H^1(0, 1) \times H^1(0, 1) \rightarrow \mathbb{C}$ defined by

$$\mathfrak{a}(v, u) := \int_0^1 \overline{v'} u' \, dx - \begin{pmatrix} \overline{v(0)} & \overline{v(1)} \end{pmatrix} B \begin{pmatrix} u(0) \\ u(1) \end{pmatrix}.$$

Let $\Delta_B: L^2(0, 1) \ni \text{dom}(\Delta_B) \rightarrow L^2(0, 1)$ denote the operator associated to \mathfrak{a} .

(a) Prove that

$$\text{dom}(\Delta_B) = \left\{ u \in H^2(0, 1) : \begin{pmatrix} -u'(0) \\ u'(1) \end{pmatrix} = B \begin{pmatrix} u(0) \\ u(1) \end{pmatrix} \right\}$$

and that $\Delta_B u$ is the second weak derivative of u for every $u \in \text{dom}(\Delta_B)$.

Hint: Use the integration by parts formula in Theorem 5.3.7(c).

(b) Show that there exist $\mu \in \mathbb{R}$ and $c > 0$ such that

$$\text{Re } \mathfrak{a}(u, u) + \mu \|u\|_{L^2(0,1)}^2 \geq c \|u\|_{H^1(0,1)}^2 \quad \text{for all } u \in H^1(0, 1).$$

Conclude that Δ_B is closed and that $s(\Delta_B) \leq \mu$.

(c) Prove that the following are equivalent:

- (i) There exists $\lambda_0 \geq s(\Delta_B)$ such that $\mathcal{R}(\lambda, \Delta_B) \geq 0$ for all $\lambda \geq \lambda_0$.
- (ii) $\mathcal{R}(\lambda, \Delta_B) \geq 0$ for all $\lambda > s(\Delta_B)$
- (iii) All off-diagonal entries of the matrix B are positive, i.e. $B_{jk} \geq 0$ for $j \neq k$.

Hint: use Stampacchia's lemma (Example 4.1.4(d)), the Beurling-Deny criterion (Theorem 5.1.7), and Theorem 5.4.1.

Notes for Chapter 5

Positivity via forms

The characterisation of positivity via forms (Theorem 5.1.7) can be extended to a much more general setting: for a closed convex set C in a Hilbert space H , a similar criterion can be given to characterise whether C is invariant under $\lambda\mathcal{R}(\lambda, A)$ for all sufficiently large λ . This is due to Ouhabaz and is, for instance, explained in his book [Ouh05].

Example 5.4.3 is, up to minor changes, taken from [DGK16a, Theorem 6.11]. The motivation for studying it in detail in this reference was an observation by Akhlil (private communication; see also [Akh18, Section 3]), who noted that the property $\mathcal{R}(\lambda, \Delta_B) \geq 0$ does not hold for all $\lambda > s(\Delta_B)$ in this example.

The maximum principle

The maximum principle in its many guises is a fundamental tool for the analysis of so-called *elliptic* and *parabolic* PDEs, and techniques based on the maximum principle have been developed to a high degree of sophistication. Two much-loved references on this subject include the books of Protter and Weinberger [PW84], and Gilbarg and Trudinger [GT01]. More recent books, which focus on nonlinear equations and reflect modern research trends, include those of Fraenkel [Fra11] and Pucci and Serrin [PS07].

The abstract perspective on the maximum principle presented in this chapter seems to be quite uncommon in the PDE literature, but not entirely without precedent. For example, linear differential operators which satisfy the positive minimum principle (Condition (1) of Theorem 5.2.1) are characterised in quite a general setting in [DL00, Chapter V, §5]. Closely related, but from a completely different perspective, is the minimum (or maximum) principle for generators of Markov processes, often attributed to Dynkin; see for example [Dyn65, Chapter V, §3] or [Sch21, Lemma 7.20].

Regularity of the boundary

In Section 5.3, we have stated results for domains $\Omega \subseteq \mathbb{R}^n$ with C^k boundary ($k \geq 1$), where C^k regularity is defined using level sets (Definition 5.3.1). It is just as reasonable to define regularity of the boundary in terms of local charts: in short, $\partial\Omega$ is of class C^k if it is locally the graph of a function $f \in C^k(\mathbb{R}^{n-1}; \mathbb{R})$. This is precisely how we defined Lipschitz

boundaries in Definition 3.B.1. The former definition is efficient, but lacks the general scope of the latter. However, for C^k regularity ($k \geq 1$), the two definitions are equivalent. The reader should not be surprised to learn that this is due to the implicit function theorem, although it is still quite some effort to present the arguments rigorously; a proof may be found, for instance, in [Hen05, Theorem 1.3].

Many properties of functions and function spaces relevant for PDE analysis depend subtly on boundary regularity. For the Sobolev embedding theorems, Lipschitz regularity is sufficient for many applications, and yields all the ‘standard’ embeddings as discussed in Supplement 5.B. For a much more thorough investigation of optimal geometric conditions and finer embedding theorems, the classic text [AF03, Chapter 4] remains an indispensable reference.

In contrast, within the class of Lipschitz domains, the elliptic regularity for the Dirichlet Laplacian (Theorem 5.3.2) holds only in special cases, and is *not* true in general. The study of regularity of PDE solutions on ‘rough’ domains is challenging and very much an active research area. A well-known reference on this subject is the monograph of Grisvard [Gri11], which has a rather ‘classical PDE’ flavour. However, much progress in this subject has also relied on techniques from harmonic analysis; see, for instance, the treatise of Kenig [Ken94].

Encore: If you want to know more...

5.A Sobolev spaces over intervals

In this section, we present some essential facts about Sobolev functions in dimension 1, beginning with a version of the fundamental theorem of calculus (Theorem 5.3.7(a) in the main text).

Theorem 5.A.1. *Let $\varnothing \neq I \subseteq \mathbb{R}$ be an open interval (not necessarily bounded), and let $p \in [1, \infty]$. If $u \in W^{1,p}(I)$, then u has a continuous representative \tilde{u} such that*

$$\tilde{u}(x) = \tilde{u}(y) + \int_y^x u'(t) dt \quad \forall x, y \in I. \quad (5.A.1)$$

The proof will be established via two lemmas of a technical nature.

Lemma 5.A.2. *Let $f \in L^1_{\text{loc}}(I)$, fix $x_0 \in I$, and define*

$$g(x) := \int_{x_0}^x f(t) dt, \quad x \in I.$$

Then $g \in C(I)$ and g is weakly differentiable with $g' = f$.

Proof. Write $I = (a, b)$ and let $\varphi \in C_c^\infty(I)$. Then

$$\int_I g(x)\varphi'(x) dx = - \int_a^{x_0} \left(\int_x^{x_0} f(t) dt \right) \varphi'(x) dx + \int_{x_0}^b \left(\int_{x_0}^x f(t) dt \right) \varphi'(x) dx.$$

We now compute the right hand side of the above equality using Fubini's theorem, which yields

$$- \int_a^{x_0} \int_a^t \varphi'(x) dx f(t) dt + \int_{x_0}^b \int_t^b \varphi'(x) dx f(t) dt = - \int_a^b \varphi(t) f(t) dt.$$

Hence $\int_I g\varphi' dx = - \int_I f\varphi dx$ for all $\varphi \in C_c^\infty(I)$, and the lemma is proved. \square

Lemma 5.A.3. *Let $f \in L^1_{\text{loc}}(I)$ satisfy*

$$\int_I f\varphi' dx = 0 \quad \forall \varphi \in C_c^\infty(I). \quad (5.A.2)$$

Then f is constant a.e. in I .

Proof. We follow [Bre11, Lemma 8.1], which features a very clever trick. Fix a function $w \in C_c^\infty(I)$ such that $\int_I w \, dx = 1$. Given $\psi \in C_c^\infty(I)$, we define

$$\varphi(x) := \int_a^x \psi(t) - \left(\int_I \psi \, ds \right) w(t) \, dt.$$

Clearly $\varphi \in C^\infty(I)$, and moreover $\varphi(a) = \varphi(b) = 0$ by construction. Since ψ, w both have compact support, it follows that $\varphi \in C_c^\infty(I)$. Hence, the assumption (5.A.2) yields

$$0 = \int_I f \varphi' \, dx = \int_I f \left[\psi - \left(\int_I \psi \, dt \right) w \right] \, dx = \int_I \psi \left[f - \left(\int_I f w \, dt \right) \right] \, dx.$$

Since $\psi \in C_c^\infty(I)$ was arbitrary, we deduce that $f - \int_I f w \, dt = 0$ a.e. in I . \square

Proof of Theorem 5.A.1. Let $u \in W^{1,p}(I)$ and fix $y \in I$. By Lemma 5.A.2, the function defined by

$$\tilde{u}(x) := \int_y^x u'(t) \, dt, \quad x \in I$$

is continuous on I and weakly differentiable with $\tilde{u}' = u'$. By definition of weak derivatives, it therefore holds that $\int_I u \varphi' \, dx = \int_I \tilde{u} \varphi' \, dx$ for all $\varphi \in C_c^\infty(I)$. However, this implies that $u - \tilde{u} = c$ a.e. in I for some constant $c \in \mathbb{C}$ by Lemma 5.A.3. We easily deduce that $C = \tilde{u}(y)$, and consequently \tilde{u} is a continuous representative of u that satisfies (5.A.1). \square

Theorem 5.A.4 (Sobolev embedding in dimension 1). *Let $p \in [1, \infty]$ and let $\emptyset \neq I \subseteq \mathbb{R}$ be an open interval. Then $W^{k,p}(I) \hookrightarrow C_b^{k-1}(I)$; more precisely, every $u \in W^{k,p}(I)$ has a representative $\tilde{u} \in C^{k-1}(I)$ such that $\partial^\alpha \tilde{u} \in C_b(I)$ for all multi-indices $|\alpha| \leq k-1$.*

If I is a bounded interval, the theorem above is a direct consequence of Theorem 5.A.1 applied to u and all weak derivatives of order $\leq k-1$. In particular, this completes the proof of Theorem 5.3.7(b) in the main text. If the interval is unbounded, it is not obvious why the continuous representative is bounded, and hence additional tools are needed. A detailed proof in this general case can be found in [Bre11, Theorem 8.8].

We conclude this section by stating a Sobolev version for the product rule and integration by parts.

Proposition 5.A.5. *Let $\emptyset \neq I = (a, b) \subset \mathbb{R}$ be a bounded interval, let $p \in [1, \infty]$, and suppose $u, v \in W^{1,p}(I)$. Then $uv \in W^{1,p}(I)$ with $(uv)' = u'v + uv'$. Moreover, the classical integration by parts formula holds:*

$$\int_y^x u' v \, dt = (u(x)v(x) - u(y)v(y)) - \int_y^x uv' \, dt, \quad \forall x, y \in [a, b].$$

Once it is known that functions in $W^{1,p}(I)$ can be approximated by functions in $C^1(\bar{I})$, then the product rule can be proved by first using the classical product rule, then passing to the limit. The integration by parts formula follows immediately from integrating the product rule. Note that the point evaluations are well-defined, thanks to Theorem 5.A.1. We refer to [Bre11, Corollary 8.10] for the full details.

5.B Sobolev embedding theorems

This section is a brief (!) overview of **Sobolev embedding theorems**, which play a fundamental role in the analysis of partial differential equations and the calculus of variations. A detailed treatment of this subject is usually part of a dedicated course in PDE theory. Thus, we do not show many proofs, but rather focus on the overall strategy and key ideas.

Before we begin, recall that a Banach space Y is said to **embed continuously** into a Banach space X , written as $Y \hookrightarrow X$, if $Y \subseteq X$ and there exists $C > 0$ such that $\|y\|_X \leq C \|y\|_Y$ for all $y \in Y$.

As a basic intuition, we may consider Sobolev spaces to be a compromise between L^p spaces and the classical spaces of continuously differentiable functions. From this perspective, given $k \in \mathbb{N}$, $p \in [1, \infty]$, and $u \in W^{k,p}(\Omega)$, there are two natural questions:

1. (*Improved integrability*) Since u is ‘better than the average’ L^p function, does it also belong to some $L^q(\Omega)$ with $q > p$?
2. (*Recovery of classical functions*) If u has sufficiently many weak derivatives, does it then have a continuous representative, or even an m -times continuously differentiable representative for some $m \in \mathbb{N}$?

Theorem 5.A.4 gives a satisfying answer in dimension 1. In higher dimensions, however, the answers to the same questions are surprisingly subtle, and depend on the precise relationship between the parameters k, p , the dimension n , and even on geometric properties of the set Ω . Fortunately, much can be learned from the simplest case $\Omega = \mathbb{R}^n$, which is already non-trivial. We begin with the Sobolev–Gagliardo–Nirenberg inequality in \mathbb{R}^n (Theorem 5.B.1) for the Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ with $1 \leq p < n$. In brief, the result asserts that a function in $W^{1,p}$ automatically has better integrability and belongs to L^q for a precisely determined $q > p$.

Theorem 5.B.1 (Sobolev, Gagliardo, Nirenberg). *Let $1 \leq p < n$, and define the **Sobolev exponent***

$$p^* := \frac{np}{n-p}.$$

Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^}(\mathbb{R}^n)$; in particular, there exists a constant $C > 0$ such that*

$$\|u\|_{p^*} \leq C \|\nabla u\|_p \quad \forall u \in W^{1,p}(\mathbb{R}^n). \quad (5.B.1)$$

This is a standard result in PDE analysis and thus can be found in many textbooks. Some classic choices include [GT01, Theorem 7.10], [Eva10, Section 5.6, Theorem 1], and [Bre11, Theorem 9.9]. The following simple argument helps to get some intuition for the result. We *suppose* for the moment that an estimate of the form

$$\|u\|_q \leq C \|\nabla u\|_p$$

holds for some $q \in (p, \infty)$ and for all $u \in C_c^\infty(\mathbb{R}^n) \subseteq W^{1,p}(\mathbb{R}^n)$. Since \mathbb{R}^n is invariant under the dilation maps $x \mapsto \lambda x$ for all $\lambda > 0$, the estimate should also hold for the dilated

functions $x \mapsto u_\lambda(x) := u(\lambda x)$. We compute

$$\frac{1}{\lambda^{n/q}} \|u\|_q = \|u_\lambda\|_q \leq C \|\nabla u_\lambda\|_p = C \lambda^{1-n/p} \|\nabla u\|_p$$

and hence

$$\|u\|_q \leq C \lambda^{1-(\frac{n}{p}-\frac{n}{q})} \|\nabla u\|_p, \quad \forall \lambda > 0.$$

In order for the right hand side of the latter inequality to remain bounded for $\lambda \rightarrow 0$ and also $\lambda \rightarrow \infty$, the exponent in λ must be 0, and hence we require $\frac{1}{n} = \frac{1}{p} - \frac{1}{q}$. Upon solving for q , we obtain $q = \frac{np}{n-p} = p^*$. The necessity of the condition $p < n$ is also evident.

Theorem 5.B.1 has a standard follow-up result.

Corollary 5.B.2. *Let $n \in \mathbb{N}$.*

- (a) *If $1 \leq p < n$, then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ for all $q \in [p, p^*]$.*
- (b) *If $n \geq 2$, then $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ for every $q \in [n, \infty)$.*

The first part of the corollary is a consequence of the following interpolation inequality, which is proved by a direct application of Hölder's inequality: if $v \in L^r(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ with $1 \leq r \leq s \leq \infty$, then $v \in L^p(\mathbb{R}^n)$ for all $p \in [r, s]$ and

$$\|v\|_p \leq \|v\|_s^{1-\theta} \|v\|_r^\theta \tag{5.B.2}$$

where $\theta \in (0, 1)$ satisfies $\frac{1}{p} = \frac{(1-\theta)}{s} + \frac{\theta}{r}$. Part (b) can be proved by recycling some steps from the proof of Theorem 5.B.1 followed by an induction argument; see e.g. [Bre11, Corollary 9.11] or [Leo09, Exercise 12.37] for details. Although it is a tempting conjecture (since $p^* \uparrow \infty$ as $p \uparrow n$), one does *not* achieve the embedding $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$, as is explained in Remark 5.B.5(a) below.

The remaining case $p > n$ is handled by the theorem of Morrey.

Theorem 5.B.3 (Morrey). *Let $n < p < \infty$. Then $u \in W^{1,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$; moreover, there exists a constant $C > 0$ such that for every $u \in W^{1,p}(\mathbb{R}^n)$, one has*

$$|u(x) - u(y)| \leq C |x - y|^{1-\frac{n}{p}} \|\nabla u\|_p \quad \text{a.e. } x, y \in \mathbb{R}^n \tag{5.B.3}$$

*In particular, every $u \in W^{1,p}(\mathbb{R}^n)$ has a bounded, continuous representative.*¹¹

Again, we refer to standard literature for the proof, for instance [Bre11, Theorem 9.12], [Eva10, Section 5.6, Theorem 4] and [GT01, Theorem 7.10].

The above results can be applied inductively to obtain embeddings for higher-order Sobolev spaces. Part (c) of the following result is especially important for PDE applications, as it establishes a relationship between weak and classical derivatives.

¹¹Readers familiar with the Hölder spaces will observe further that the continuous representative belongs to $C^{0,\alpha}(\mathbb{R}^n)$ with $\alpha = 1 - n/p$.

Corollary 5.B.4 (Summary of Sobolev embeddings in \mathbb{R}^n). *Let $k, n \in \mathbb{N}$ with $n \geq 2$ and $1 \leq p < \infty$. One has the following continuous embeddings:*

- (a) *If $kp < n$, then $W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ for all $q \in [p, p^*(k)]$ where $p^*(k) := \frac{np}{n-kp}$.*
- (b) *If $kp = n$, then $W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ for all $q \in [p, \infty)$.*
- (c) *If $kp > n$ and $m \in \mathbb{N}_0$ satisfies $m < k - \frac{n}{p}$, then $W^{k,p}(\mathbb{R}^n) \hookrightarrow C_b^m(\mathbb{R}^n)$; more precisely, every $u \in W^{k,p}(\mathbb{R}^n)$ has a representative $\tilde{u} \in C^m(\mathbb{R}^n)$ such that $\partial^\alpha \tilde{u} \in C_b(\mathbb{R}^n)$ for all multi-indices $|\alpha| \leq m$.*

Proof. We provide the details, since they are often omitted in PDE books. For brevity, we write D to denote any first-order partial derivative and omit writing \mathbb{R}^n from the spaces.

- (a) Theorem 5.B.1 yields the result for $k = 1$. Now assume that the claim holds for $kp < n$ for some $k \geq 1$, and let $u \in W^{k+1,p}$ with $(k+1)p < n$. Then $u, Du \in W^{k,p} \hookrightarrow L^{p^*(k)}$ by the induction hypothesis. Since $(k+1)p < n$ implies $p^*(k) < n$, therefore $u \in W^{1,p^*(k)} \hookrightarrow L^{(p^*(k))^*} = L^{p^*(k+1)}$ by Theorem 5.B.1, and the inductive step is complete. We then obtain $W^{k,p} \hookrightarrow L^q$ for all $q \in [p, p^*(k)]$ by the interpolation inequality (5.B.2).
- (b) Corollary 5.B.2 covers the case of $k = 1$, and hence we may assume that $u \in W^{k, \frac{n}{k}}$ for some $k \geq 2$. We have $u, Du \in W^{k-1, \frac{n}{k}} \hookrightarrow L^{(n/k)^*(k-1)} = L^n$ by part (a), since $(k-1)\frac{n}{k} < n$. Thus $u \in W^{1,n} \hookrightarrow L^q$ for all $q \in [p, \infty)$ by Corollary 5.B.2.
- (c) Write $k = m + \ell$, where $\ell \in \mathbb{N}$ satisfies $\ell - 1 \leq \frac{n}{p} < \ell$. We treat the case $m = 0$ first, so $k = \ell$. Then $u, Du \in W^{(k-1)p}$ where $(k-1)p \leq n$. If $(k-1)p < n$, it follows from part (a) that $W^{(k-1)p} \hookrightarrow L^r$, where $r = p^*(k-1) > n$. Consequently $u \in W^{1,r}$ with $r > n$, so Theorem 5.B.3 yields that $u \in C_b$. If $(k-1)p = n$, then part (b) yields that $W^{(k-1)p} \hookrightarrow L^r$ for all $r \in [p, \infty)$. By choosing $r > p \vee n$, we obtain from Theorem 5.B.3 again that $u \in W^{1,r} \hookrightarrow C_b$.

In the general case $m \geq 1$, the above conclusion holds for $\partial^\alpha u$ for all multi-indices $|\alpha| \leq m$, and thus $u \in C_b^m$. \square

Let us comment on some exceptional cases to highlight some of the many hidden subtleties in Sobolev embedding theorems.

Remarks 5.B.5.

- (a) It is important to note that Corollary 5.B.2 does *not* assert that $W^{1,n}(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n)$. For $n \geq 2$, this is false! Here is a standard example: define a function $u \in C^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{R})$ such that

$$u(x) = \log \left(\log \left(\frac{4}{\|x\|_2} \right) \right), \quad \|x\|_2 < 1,$$

and $u(x) = 0$ for all $\|x\|_2 \geq 2$. Clearly $u \notin L^\infty(\mathbb{R}^n)$, due to the singularity at $x = 0$. It is then a slightly tedious exercise to check that $u \in W^{1,n}(\mathbb{R}^n)$ nevertheless.

In fact, by further and even more tedious calculations, the same example works to show that $W^{k,p}(\mathbb{R}^n) \not\subset L^\infty(\mathbb{R}^n)$ if $kp = n$ and $p > 1$; see [AF03, Example 4.43].

- (b) The case $p = 1$ and $k = n$ is special: we have $W^{n,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ (and hence $W^{n,1}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$ by interpolation). Indeed, if $u \in C_c^\infty(\mathbb{R}^n)$ and $x = (x_1, \dots, x_n)$ is arbitrary, we have

$$u(x) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} (\partial_1 \partial_2 \cdots \partial_n u)(t_1, \dots, t_n) dt_1 \cdots dt_n,$$

and hence

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq \|\partial_1 \partial_2 \cdots \partial_n u\|_{L^1(\mathbb{R}^n)} \leq \|u\|_{W^{n,1}(\mathbb{R}^n)}.$$

The inequality for general $u \in W^{n,1}(\mathbb{R}^n)$ follows, as usual, by taking an approximating sequence of test functions (u_k) such that $u_k \rightarrow u$ in $W^{n,1}$ and $u_k(x) \rightarrow u(x)$ a.e. $x \in \mathbb{R}^n$, cf. Corollary 3.A.5.

- (c) Theorem 5.B.3 does not extend to $p = \infty$, since the typical proofs require the approximation result Corollary 3.A.5. Nevertheless, it is true that a function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ is (uniformly) Lipschitz continuous if and only if $u \in W^{1,\infty}(\mathbb{R}^n)$; a proof may be found, for instance, in [Eva10, Theorem 4, Section 5.8].

Corollary 5.B.4 gives a rather satisfactory picture of the Sobolev embeddings in \mathbb{R}^n . We now turn to the case of domains, i.e. open subsets $\Omega \subseteq \mathbb{R}^n$, where the situation is not so neat. It turns out that the results are highly influenced by geometric properties of the boundary $\partial\Omega$, and many different approaches are available for the analysis of such properties. Here, we choose to present the *extension technique*, which can be formulated easily in operator-theoretic terms (see Theorem 5.B.7 below).

The key idea is conceptually quite simple.¹² Given a non-empty open subset $\Omega \subseteq \mathbb{R}^n$, we ask if there is a method to extend each function $u \in W^{k,p}(\Omega)$ to some $\tilde{u} \in W^{k,p}(\mathbb{R}^n)$, for which the embedding theorems in \mathbb{R}^n can then be applied. Any such method must necessarily take into account the local behaviour of u near the boundary $\partial\Omega$, in order to preserve the values of u within Ω while allowing weak differentiability on \mathbb{R}^n . The following example shows that Ω cannot be arbitrary if this is to work.

Example 5.B.6 (cf. [Maz11, Example 2, Section 1.5.1]). Consider the set

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < x_1^3\}$$

and define $u(x) := x_1^{1-\beta}$ with $1 < \beta < 2$. By elementary calculations and using the result of Exercise 3.4(b), one checks that $u \in H^1(\Omega) = W^{1,2}(\Omega)$.

Suppose that u can be extended to a function $\tilde{u} \in H^1(\mathbb{R}^2)$. By Corollary 5.B.2(b), it holds that $\tilde{u} \in L^q(\mathbb{R}^2)$ for all $q \geq 2$. But then

$$\int_{\mathbb{R}^2} |\tilde{u}(x)|^q dx \geq \int_{\Omega} |u(x)|^q dx = \int_0^1 \int_0^{x_1^3} x_1^{(1-\beta)q} dx_2 dx_1 = \int_0^1 x_1^{3-(\beta-1)q} dx_1,$$

¹²But technically quite involved!

and we can choose $q > 2$ sufficiently large so that the final integral does not converge. This is a contradiction, and thus we deduce that no such extension \tilde{u} exists.

The previous example illustrates what happens when the boundary $\partial\Omega$ is badly behaved: at the point $(0,0)$, the set Ω has a **cusp**, which allows functions to blow-up while still being in a Sobolev space. However, for the extended function, the boundary is ‘removed’ and membership in a Sobolev space now excludes very wild behaviour.

It turns out that Lipschitz boundary (see Definition 3.B.1) is the sweet spot for the extension technique.

Theorem 5.B.7 (Sobolev extension operator). *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. For every $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, there exists a bounded linear operator $E: W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)$ such that $(Eu)(x) = u(x)$ for a.e. $x \in \Omega$. In particular, there exists a constant $C > 0$ such that*

$$\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{k,p}(\Omega)} \quad (5.B.4)$$

for $u \in W^{k,p}(\Omega)$.

There is a large variety of techniques for creating extension operators; see e.g. [AF03, Section 5.4] and the references therein. In the statement of Theorem 5.B.7, the operator E a priori depends on the parameters k and p , as well as Ω of course. However, a remarkable theorem of Elias Stein shows that in fact there is a *universal* construction, in the sense that given a bounded domain with Lipschitz boundary, there is an extension operator E that works simultaneously for all $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. We refer the reader to Stein’s original monograph [Ste70, Chapter 6, §3] or [Leo09, Theorem 13.17] for the details of this beautiful result.

Using the heavy machinery of Theorem 5.B.7, it is now almost trivial to write down Sobolev embeddings for bounded Lipschitz domains. We highlight in particular the embedding into C^m spaces, since it is used in the main text.

Theorem 5.B.8. *Let $n \geq 2$ and let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then all the embeddings of Corollary 5.B.4 remain true when \mathbb{R}^n is replaced by Ω . In particular, if $kp > n$ and $m \in \mathbb{N}_0$ satisfies $m < k - \frac{n}{p}$, then $W^{k,p}(\Omega) \hookrightarrow C^m(\overline{\Omega})$.*

Proof. If $u \in W^{k,p}(\Omega)$, then by Corollary 5.B.4 and Theorem 5.B.7 we obtain

$$\|u\|_{L^q(\Omega)} \leq \|Eu\|_{L^q(\mathbb{R}^n)} \leq C \|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq \tilde{C} \|u\|_{W^{k,p}(\Omega)}$$

in the case $kp \leq n$, where \tilde{C} is the product of C in (5.B.4) with the implied embedding constants in Corollary 5.B.4. Likewise, in the case $kp > n$ we obtain $\|u\|_{C^m(\overline{\Omega})} \leq \tilde{C} \|u\|_{W^{k,p}(\Omega)}$. \square

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