Chapter 4

Ordered function spaces and Banach lattices

After studying (eventual) positivity for matrices in Chapters 1 and 2, we encountered a selection of unbounded operators in infinite-dimensional spaces in Chapter 3. It is our goal to analyse positivity properties related to such operators. In particular, the question of when the resolvent of an unbounded operator is positive will be important.

This requires an order structure on the underlying Banach spaces. All examples in Chapter 3 were defined on function spaces, where it is natural to consider the pointwise (almost everywhere) order. Yet, it turns out that this order behaves quite differently on, for instance, L^p-spaces compared to $C_0(\Omega)$. In this chapter, we introduce the general framework of **Banach lattices** that includes many classical function spaces and allows us to develop the subsequent theory on all such spaces simultaneously.[1]

4.1 Real Banach lattices and function spaces

We recall a few elementary notions about partially ordered sets. Consider a set X together with a **partial order** $\le$ on X, which means that $\le$ is a reflexive, anti-symmetric and transitive relation on X. Let $S \subseteq X$. An element $b \in X$ is called an **upper bound** of S if $x \le b$ for all $x \in S$, and b is called a **lower bound** of S if $b \le x$ for all $x \in S$. The element $b \in X$ is called the **least upper bound** or the **supremum** of S if b is an upper bound of S and, in addition, $b \le c$ for all upper bounds c of S.[2] Similarly, b is called the **greatest lower bound** or the **infimum** of S if b is a lower bound of S and, in addition, $b \ge c$ for all lower bounds of S. It is not difficult to see that the supremum or infimum of a set is unique if it exists. We denote the supremum and the infimum of S respectively by $\sup S$ and $\inf S$ whenever they exist. Finally, for two elements $x, y \in X$ we use the notation $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ whenever they exist.

[1] Even beyond a unified treatment of different types of function spaces, there are good reasons to develop the theory in the general setting of Banach lattices. We explain this later on.

[2] In other words, b is an upper bound of S and a lower bound of the set of all upper bounds.

Definition 4.1.1 (Real vector lattices)**.** A **real vector lattice**[3] is a real vector space V together with a partial order $\leq$ on V that satisfies the following properties:

(I) The order $\leq$ is compatible with the linear structure of V, i.e. for all $x, y, z \in V$ that satisfy $x \leq y$ and all real numbers $\alpha \in [0, \infty)$ one has $x + z \leq y + z$ and $\alpha x \leq \alpha y$.

(II) Any two elements $x, y \in V$ have a supremum $x \vee y$ in V.

A vector subspace W of V is called a **vector sublattice** of V if $x \vee y \in W$ for all $x, y \in W$.

One can easily deduce from the definition of a vector lattice V that for all $x, y \in V$ the infimum $x \wedge y$ also exists and is equal to $-\big((-x) \vee (-y)\big)$. We frequently use the following concepts in vector lattices.

Definition 4.1.2. Let V be a real vector lattice.

(a) The set $V_+ := \{x \in V : x \geq 0\}$ is called the **positive cone** or, briefly, the **cone** of V.

The elements of V_+ are called the **positive** elements of V.

(b) For all $x \in V$, we call the elements

$$x^+ := x \vee 0, \qquad x^- := (-x)^+, \qquad |x| := x \vee (-x)$$

of V_+ the **positive part**, the **negative part**, and the **modulus** of x respectively.[4]

Observe that for elements x, y of a vector lattice V, one has $x \leq y$ if and only $y - x \in V_+$. Moreover, for each $x \in V$ the negative part is given by $x^- = -(x \wedge 0)$.

Proposition 4.1.3 (Algebraic properties in vector lattices)**.** *Let x, y, z be elements of a real vector lattice V. The following identities hold:*

(a) $x \vee y = -\big((-x) \wedge (-y)\big)$.

(b) *Distributive law:* $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ *and* $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

(c) *Translation property:* $(x \vee y) + z = (x + z) \vee (y + z)$ *and* $(x \wedge y) + z = (x + z) \wedge (y + z)$.

(d) *Scaling property:* $\alpha(x \vee y) = (\alpha x) \vee (\alpha y)$ *and* $\alpha(x \wedge y) = (\alpha x) \wedge (\alpha y)$ *for all scalars* $\alpha > 0$, *and* $|\alpha x| = |\alpha|\,|x|$ *for all scalars* $\alpha \in \mathbb{R}$.

(e) *Triangle inequalities:* $\big|x + y\big| \leq |x| + \big|y\big|$ *and* $\big|x - y\big| \geq \big||x| - |y|\big|$.

(f) $x \vee y = \frac{1}{2}\big(x + y + |x - y|\big)$ *and* $x \wedge y = \frac{1}{2}\big(x + y - |x - y|\big)$.

(g) $x = x^+ - x^-$ *and* $|x| = x^+ + x^-$.

Proof. (a)–(e) These are straightforward to verify directly from the definitions, so we omit the details.

[3] Also known as a **Riesz space**, especially in the Dutch tradition.

[4] One can also call $|x|$ the **absolute value** of x.

(f) Using the definition of the modulus and properties (c) and (d), observe that

$$x+y+\left|x-y\right| = x+y+(x-y)\vee(y-x) = (2x)\vee(2y) = 2(x\vee y).$$

Similarly, using (a) in addition, we find

$$\begin{aligned} x+y-\left|x-y\right| &= x+y-[(x-y)\vee(y-x)] \\ &= x+y+(y-x)\wedge(x-y) = (2y)\wedge(2x) = 2(x\wedge y). \end{aligned}$$

(g) By setting $y=0$ in part (f), we obtain

$$x^+ = x\vee 0 = \tfrac{1}{2}(x+|x|) \quad \text{and} \quad x^- = (-x)\vee 0 = -(x\wedge 0) = -\tfrac{1}{2}(x-|x|).$$

Adding and subtracting these yields the claim. □

Examples 4.1.4. We discuss some standard examples of vector lattices.

(a) We consider $\mathbb{R}^n$ with the componentwise ordering introduced in Chapter 1. It is easy to check that the supremum of any two vectors exists and is given by

$$(x\vee y)_i = x_i \vee y_i \quad \forall\, x,y\in\mathbb{R}^n,\, i=1,\dots,n;$$

where $\vee$ on the right-hand side is simply the usual supremum in $\mathbb{R}$, and likewise for the infimum. With these operations, it is clear that $\mathbb{R}^n$ is a real vector lattice. Equivalently, one can use the definition of the modulus introduced in Definition 1.1.4 and recover the lattice operations via the identities in Proposition 4.1.3(f), which of course hold for real numbers.

(b) The space $\mathrm{C}(\Omega;\mathbb{R})$ of continuous real-valued functions defined on $\emptyset\neq\Omega\subseteq\mathbb{R}^n$ is a real vector lattice with the natural pointwise ordering and lattice operations, i.e. $f\le g$ if and only if $f(x)\le g(x)$ for all $x\in\Omega$, $\left|f\right|(x) := \left|f(x)\right|$, and so on.

(c) Let (Ω,μ) be a measure space. The Lebesgue spaces $L^p(\Omega,\mu;\mathbb{R})$, $p\in[1,\infty]$ admit a partial order $f\le g$ if and only if $f(\omega)\le g(\omega)$ for μ-a.e. ω and a modulus $\left|f\right|(\omega) := \left|f(\omega)\right|$. The lattice operations are again expressed via the identities in Proposition 4.1.3(f), and thus we obtain a vector lattice structure on $L^p(\Omega,\mu;\mathbb{R})$.

(d) Now for a perhaps slightly surprising example: For each open set $\emptyset\neq\Omega\subseteq\mathbb{R}^n$ and each $p\in[1,\infty]$, the Sobolev space $W^{1,p}(\Omega;\mathbb{R})$ is a vector sublattice of $L^p(\Omega;\mathbb{R})$ and

$$\partial_k(u^+) = \mathbb{1}_{\{u>0\}}\,\partial_k u, \qquad \partial_k(u^-) = -\,\mathbb{1}_{\{u<0\}}\,\partial_k u, \qquad \partial_k(|u|) = \operatorname{sgn}(u)\,\partial_k u$$

for every $u\in W^{1,p}(\Omega;\mathbb{R})$ and every $k\in\{1,\dots,n\}$. This is a famous result due to the Italian mathematician Guido Stampacchia, often referred to as the **Stampacchia lemma**. To avoid a detour, we do not prove this result here. Readers interested in the details can find a proof in Supplement 4.B.

(e) If $p \neq \infty$, then the Sobolev space $W_0^{1,p}(\Omega;\mathbb{R})$ (recall that we did not define $W_0^{1,\infty}$) is a vector sublattice of $W^{1,p}(\Omega;\mathbb{R})$ (and thus of $L^p(\Omega;\mathbb{R})$ by (d)). The intuition here is quite clear: we think of $W_0^{1,p}(\Omega;\mathbb{R})$ as the subspace of those $u \in W^{1,p}(\Omega;\mathbb{R})$ that vanish at the boundary $\partial\Omega$. If u vanishes at the boundary, it seems plausible that $|u|$ does as well. A rigorous proof requires more theory of Sobolev spaces, so we do not show it in the lectures. Readers interested in the details can find them in [AU23, Theorem 6.37 & 6.39].

We now specialise to vector lattices which are also Banach spaces, so that all the standard tools of functional analysis are available at our disposal.

Definition 4.1.5 (Real Banach lattices)**.** A **real Banach lattice** is a Banach space E over $\mathbb{R}$ with a partial order $\le$ that turns E into a real vector lattice that is compatible with the norm in the following sense: whenever $x, y \in E$ satisfy $|x| \le |y|$, then $\|x\| \le \|y\|$.

Norms on real vector lattices satisfying above compatibility condition are called **lattice norms**. Note that instead of the assumption that $|x| \le |y|$ implies $\|x\| \le \|y\|$, one could equivalently require that the following two properties are satisfied:

(a) For all $x \in E$ one has $\|x\| = \||x|\|$.

(b) For all $x, y \in E$ the inequalities $0 \le x \le y$ imply $\|x\| \le \|y\|$.

Let us now discuss which of the vector lattices in Examples 4.1.4 are Banach lattices.

Examples 4.1.6.

(a) For a given $p \in [1,\infty]$, it is easy to verify that the ℓ^p norm on $\mathbb{R}^n$, defined by

$$\|x\|_p := \begin{cases} \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} & \text{if } 1 \le p < \infty \\ \max_{k=1,\dots,n} |x_k| & \text{if } p = \infty, \end{cases}$$

is a lattice norm on $\mathbb{R}^n$. Since $\mathbb{R}^n$ is complete with respect to any of the above norms, it follows that $(\mathbb{R}^n, \|\cdot\|_p)$ is a real Banach lattice.

However, one can also construct norms on $\mathbb{R}^n$ which are not lattice norms. For instance, if $n = 2$, consider the norm

$$\|x\|_W := |x_1| + |x_2 - x_1|.$$

Observe that the vectors $a = (1,-1)^{\mathrm{T}}$ and $b = (1,1)^{\mathrm{T}}$ have the same modulus, but $\|a\|_W = 3 > \|b\|_W = 1$.

(b) If $\Omega = K$ is compact, then clearly the supremum norm $\|f\|_\infty := \sup_{x\in K} |f(x)|$ is a lattice norm on $\mathrm{C}(K;\mathbb{R})$. As $\mathrm{C}(K;\mathbb{R})$ is complete with respect to the supremum norm, it is a real Banach lattice.

(c) It is also clear that the L^p norms are lattice norms, and thus the $L^p(\Omega, \mu; \mathbb{R})$ spaces are real Banach lattices.

(d) The $W^{1,p}$ norms are *not* lattice norms. Since the $W^{1,p}$ norm involves the (weak) partial derivatives of functions, it is easy to see that it does not satisfy the lattice norm property (e.g. one can consider smooth, bounded functions with oscillations). Thus the Sobolev spaces $W^{1,p}(\Omega; \mathbb{R})$ are examples of real vector lattices which are Banach spaces but not real Banach lattices.

As the definition suggests, the setting of a Banach lattice allows order, lattice, and topological structures to fit together in a useful way. Let us makes this precise.

Proposition 4.1.7. *Let E be a real Banach lattice.*

(a) *The mappings $x \mapsto |x|$, $x \mapsto x^+$, and $x \mapsto x^-$ are continuous from E to E.*

(b) *The mappings $(x, y) \mapsto x \vee y$ and $(x, y) \mapsto x \wedge y$ are continuous from $E \times E$ to E.*

(c) *The positive cone E_+ is closed.*

(d) *If two sequences (x_n) and (y_n) in E converge to points $x, y \in E$, respectively such that $x_n \leq y_n$ for all indices n, then $x \leq y$.*

Proof. (a) The continuity of $|\cdot|$ is an immediate consequence of the reverse triangle inequality (Proposition 4.1.3(e)) and the fact that the norm is a lattice norm (Definition 4.1.5). The continuity of the positive and negative parts is then immediate, since $x^+ = \frac{1}{2}(x + |x|)$ and $x^- = \frac{1}{2}(|x| - x)$ for all $x \in E$ (Proposition 4.1.3(g)).

(b) This follows from (a) and the representation formula of $\vee$ and $\wedge$ in Proposition 4.1.3(f).

(c) If $(x_n) \subset E_+$ is a sequence converging to $x \in E$, then $x_n^- = 0$ for all $n \in \mathbb{N}$, and thus $x^- = \lim_{n\to\infty} x_n^- = 0$ by (a). Hence, $x \geq 0$.

(d) This an immediate consequence of (c). ☐

4.2 Complex Banach lattices

In all the previous chapters, spectral theory has been a recurring theme, and for this reason, it is important to consider vector spaces over $\mathbb{C}$. The theory of vector lattices, introduced in Section 4.1, is however a theory over the real field. To solve this issue we now study how to *complexify* a real vector space and, in a particular, a Banach lattice.

Definition 4.2.1 (Complexification of a real vector space)**.** Let V be a real vector space. We give the Cartesian product $V \times V$ the structure of a complex vector space by defining:

(i) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ for all $(x_1, y_1), (x_2, y_2) \in V \times V$;

(ii) $(\alpha + \mathrm{i}\beta)(x, y) = (\alpha x - \beta y, \beta x + \alpha y)$ for all $(x, y) \in V \times V$ and $\alpha, \beta \in \mathbb{R}$.

The resulting vector space is called the **complexification** of V, and is denoted as $V_{\mathbb{C}}$ or $V + \mathrm{i}V$. Furthermore, given $(x, y) \in V_{\mathbb{C}}$, we define **complex conjugation** by $\overline{(x, y)} := (x, -y)$.

Via the injective $\mathbb{R}$-linear map $V \to V_{\mathbb{C}}$, $x \mapsto (x, 0)$, we identify V with an $\mathbb{R}$-linear subspace of $V_{\mathbb{C}}$. Thus, $V_{\mathbb{C}}$ becomes the direct sum (over $\mathbb{R}$) of its subspaces V and $\mathrm{i}V$. As a consequence, every $z \in V_{\mathbb{C}}$ can be written uniquely in the form $z = x + \mathrm{i}y$ with $x, y \in V$. We make this explicit in the following.

Notation 4.2.2. In the setting of Definition 4.2.1, we denote each $z = (x, y) \in V_{\mathbb{C}}$ by

$$z = x + \mathrm{i}y.$$

The **real** and **imaginary parts** of z are defined by $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$, respectively. In particular, $\overline{z} = x - \mathrm{i}y$.

For a real Banach lattice E it is natural to extend the modulus function $|\cdot| : E \to E_+$ to the complexification $E_{\mathbb{C}}$, just as the modulus on $\mathbb{R}$ can be extended to $\mathbb{C}$. We deviate from the standard construction in the literature and follow a more axiomatic approach.

Definition 4.2.3 (Complex modulus function)**.** Let E be a real Banach lattice and let $E_{\mathbb{C}}$ be its vector space complexification. A **complex modulus function** is a function $|\cdot|_{\mathbb{C}} : E_{\mathbb{C}} \to E_+$ that satisfies the following axioms:

(I) **Compatibility with the real modulus**: For all $x \in E$ one has $|x|_{\mathbb{C}} = |x|$.

(II) **Triangle inequality:** For all $z, w \in E_{\mathbb{C}}$ one has $|z + w|_{\mathbb{C}} \le |z|_{\mathbb{C}} + |w|_{\mathbb{C}}$.

(III) **Absolute homogeneity:** For all $z \in E_{\mathbb{C}}$ and all $\alpha \in \mathbb{C}$ one has $|\alpha z|_{\mathbb{C}} = |\alpha|\,|z|_{\mathbb{C}}$.

To obtain the existence of a modulus function on the complexification of a Banach lattice, one takes inspiration from the following formulae in $\mathbb{C}$. Given $z \in \mathbb{C}$, one can write z in the polar form $z = e^{i\varphi}|z|$ for a number $\varphi \in [0, 2\pi)$. On one hand, this gives

$$|z| = \sup_{\theta \in [0,2\pi]} \operatorname{Re}(e^{i\theta} z). \tag{4.2.1}$$

and on the other,

$$\int_0^{2\pi} \left|\operatorname{Re}(e^{i\theta} z)\right| \,\mathrm{d}\theta = |z| \int_0^{2\pi} \left|\operatorname{Re} e^{i(\varphi+\theta)}\right| \,\mathrm{d}\theta = |z| \int_0^{2\pi} |\cos(\theta)| \,\mathrm{d}\theta = 4\,|z|, \tag{4.2.2}$$

where have we used the 2π-periodicity of cos for the second equality. As only the modulus of real numbers occurs in (4.2.2), the formula shows us a potential way to construct a complex modulus function on a Banach lattice.

Theorem 4.2.4 (Existence and uniqueness of the complex modulus)**.** *Let E be a real Banach lattice and $E_{\mathbb{C}}$ its vector space complexification. There exists precisely one complex modulus function $|\cdot|_{\mathbb{C}} : E_{\mathbb{C}} \to E_+$. Moreover, it can be represented by the formulae*

$$|z|_{\mathbb{C}} = \frac{1}{4} \int_0^{2\pi} \left|\operatorname{Re}(e^{i\theta} z)\right| \,\mathrm{d}\theta = \sup_{\theta \in [0,2\pi]} \operatorname{Re}(e^{i\theta} z)$$

for all $z \in E_{\mathbb{C}}$.

The supremum in the theorem is understood within the partially ordered vector space E (note that it is not at all obvious that this supremum exists) and the integral can either be interpreted as a vector-valued Riemann integral or as a Bochner integral. Since we will often need integrals of functions with values in Banach spaces, we provide a brief overview of them in Appendix 4.A for readers who are not yet familiar with this concept.

We proof the existence result in Theorem 4.2.4 by showing that the integral formula in the theorem defines a complex modulus function. The proof of the uniqueness and of the supremum formula require more advanced tools from Banach lattice theory. We thus not prove it here and refer to [MW74, Theorem 2.2] instead.[5]

Proof of the existence and the integral formula in Theorem 4.2.4. For every $z \in E_{\mathbb{C}}$ we define $|z|_{\mathbb{C}} := \frac{1}{4}\int_0^{2\pi} \left|\mathrm{Re}(e^{i\theta}z)\right| \,\mathrm{d}\theta$ for all $z \in \mathbb{C}$. Observe that the integral exists since $\theta \mapsto \left|\mathrm{Re}(e^{i\theta}z)\right| = |\cos\theta \,\mathrm{Re}\, z - \sin\theta\, \mathrm{Im}\, z|$ is continuous from $[0, 2\pi]$ to E. We now prove that $|\cdot|_{\mathbb{C}}$ satisfies the axioms from Definition 4.2.3.

(I) For every $x \in E$ one has

$$|x|_{\mathbb{C}} = \frac{1}{4}\int_0^{2\pi} \left|\mathrm{Re}(e^{i\theta}x)\right| \,\mathrm{d}\theta = \frac{1}{4}\int_0^{2\pi} |\cos\theta|\,|x| \,\mathrm{d}\theta = |x|\,.$$

(II) This follows readily from the triangle inequality of the modulus on the real Banach lattice E (Proposition 4.1.3(e)).

(III) This follows from the same substitution that we used in formula (4.2.2). □

Given the existence and uniqueness of the complex modulus we can now define complex Banach lattices. Instead of using the notation E and $E_{\mathbb{C}}$ from above, it is often more convenient to write $E_{\mathbb{R}}$ and E. Moreover, since the complex modulus is consistent with the real one by definition, we will simply use the notation $|\cdot|$ from now on.

Definition 4.2.5 (Complex Banach lattices)**.**

(a) A **complex Banach lattice** E is a vector space complexification of a real Banach lattice $E_{\mathbb{R}}$ together with a complex modulus function $|\cdot| : E \to (E_{\mathbb{R}})_+$ and the norm $\|\cdot\|$ on E given by $\|z\| := \||z|\|_{E_{\mathbb{R}}}$.

The real Banach lattice $E_{\mathbb{R}}$ is called the **real part** of E.

(b) By a **Banach lattice** we mean a real or complex Banach lattice.

It is straightforward to check that the norm on a complex Banach lattice is indeed a norm and that it is complete, i.e. a complex Banach lattice is indeed a Banach space. By the definition of a real Banach lattice one has $\|x\|_{E_{\mathbb{R}}} = \||x|\|_{E_{\mathbb{R}}} = \|x\|$ for every $x \in E_{\mathbb{R}}$, so we can actually denote the norms on E and $E_{\mathbb{R}}$ by the same symbol $\|\cdot\|$.

[5] Unfortunately this paper is only available in German and it seems that its results have not appeared in a book. . . yet.

Remark 4.2.6 (Inequalities and the cone in a complex Banach lattice)**.** Let E be a complex Banach lattice. Its **positive cone** is defined as $E_+ := (E_\mathbb{R})_+$. If we write $x \le y$ for $y \ge x$ for two vectors $x, y \in E$, we mean tacitly that $x, y \in E_\mathbb{R}$.

Before we discuss some examples, let us note the following properties of the modulus in a complex Banach lattice.

Proposition 4.2.7. *Let E be a complex Banach lattice. Then one has $|\overline{z}| = |z|$ as well as $|\mathrm{Re}\, z| \le |z|$ and $|\mathrm{Im}\, z| \le |z|$ for all $z \in E$.*

Proof. Let $z \in E$. For all $\theta \in \mathbb{R}$ one has $\mathrm{Re}(e^{\mathrm{i}\theta}\overline{z}) = \mathrm{Re}\left(\overline{e^{-\mathrm{i}\theta} z}\right) = \mathrm{Re}(e^{-\mathrm{i}\theta} z)$, so it follows from the integral formula for $|z|$ in Theorem 4.2.4 that

$$|\overline{z}| = \frac{1}{4}\int_0^{2\pi} \left|\mathrm{Re}(e^{-\mathrm{i}\theta} z)\right| \,\mathrm{d}\theta = \frac{1}{4}\int_0^{2\pi} \left|\mathrm{Re}(e^{\mathrm{i}\varphi} z)\right| \,\mathrm{d}\varphi = |z|,$$

where the second equality uses the substitution $\varphi := 2\pi - \theta$. By using the axioms of the complex modulus (Definition 4.2.3) we thus get

$$2\,|\mathrm{Re}\, z| = |z + \overline{z}| \overset{\text{(II)}}{\le} |z| + |\overline{z}| = 2\,|z|,$$

and therefore, by axiom (III), $|\mathrm{Im}\, z| = |\mathrm{Re}(-\mathrm{i}z)| \le |-\mathrm{i}z| = |z|$. □

Examples 4.2.8.

(a) Let K be a compact metric (or topological) space. Endowed with the pointwise modulus function and the sup norm $\|\cdot\|_\infty$, $\mathrm{C}(K) = \mathrm{C}(K;\mathbb{C})$ is a complex Banach lattice with real part $\mathrm{C}(K;\mathbb{R})$.

Indeed, $\mathrm{C}(K)$ is the vector space complexification (as in Definition 4.2.1) of $\mathrm{C}(K;\mathbb{R})$ and the pointwise modulus function obviously satisfies the axioms from Definition 4.2.3. Finally, one clearly has $\|f\|_\infty = \||f|\|_\infty$.

(b) Let (Ω, μ) be a measure space and $1 \le p \le \infty$. Similarly as in (a) one can readily see that $L^p(\Omega,\mu) = L^p(\Omega,\mu;\mathbb{C})$ is a complex Banach lattice with real part $L^p(\Omega,\mu;\mathbb{R})$.

We close this section by introducing the following property of linear operators.

Definition 4.2.9 (Real operators)**.** Let E, F be complex Banach lattices. Given a linear operator $A\colon E \supseteq \mathrm{dom}\,(A) \to F$, set $\mathrm{dom}\,(A)_\mathbb{R} := \mathrm{dom}\,(A) \cap E_\mathbb{R}$. Then A is called **real** if

$$\mathrm{dom}\,(A) = \mathrm{dom}\,(A)_\mathbb{R} + i\,\mathrm{dom}\,(A)_\mathbb{R} \quad \text{and} \quad A(\mathrm{dom}\,(A)_\mathbb{R}) \subseteq F_\mathbb{R}.$$

Observe that if A is everywhere defined, then A is real if and only if $A(E_\mathbb{R}) \subseteq F_\mathbb{R}$. Typical examples of unbounded real operators are differential operators with real coefficients that are defined on Sobolev spaces.

4.3 Positive operators

It follows from Proposition 1.1.3 that the following definition generalises positivity of matrices to linear operators.

Definition 4.3.1 (Positive operators)**.**

(a) A linear map $T\colon V \to W$ between two real vector lattices V and W is called **positive** if $TV_+ \subseteq W_+$.

(b) Similarly, a linear map $T\colon E \to F$ between two complex Banach lattices is called **positive** if $TE_+ \subseteq F_+$.

Note that part of (a) of the definition includes operators between real Banach lattices, but it sometimes convenient to have the notion "positive operator" available in the more general case of vector lattices. We distinguished the cases (a) and (b) in the definition since we did not define the general concept of **complex vector lattices**.

Observe that a positive linear operator T preserves inequalities, i.e. if $x \le y$, then also $Tx \le Ty$. If the scalar field is complex, every positive operator is real, since the positive cone in a complex Banach lattice E spans the real part $E_{\mathbb{R}}$.

The following generalises Proposition 1.1.5 from matrices to positive operators.

Proposition 4.3.2. *Let E, F be Banach lattices over the same scalar field. For each positive linear operator $T \in \mathcal{L}(E,F)$ and each $x \in E$ one has $|Tx| \le T|x|$.*

Proof. First assume that the scalar field is $\mathbb{R}$. For each $x \in E$ it then follows from $\pm x \le |x|$ and the positivity of T that $\pm Tx \le T|x|$, so $|Tx| = Tx \vee (-Tx) \le T|x|$.

Now let the scalar field be $\mathbb{C}$. By applying the real case to the restriction $T|_{E_{\mathbb{R}}}$ one gets $|Tx| \le T|x|$ for every $x \in E_{\mathbb{R}}$. For the general case $x \in E$ we thus have

$$|Tx| = \frac{1}{4}\int_0^{2\pi} \left|\mathrm{Re}(e^{i\theta}Tx)\right| \,\mathrm{d}\theta = \frac{1}{4}\int_0^{2\pi} \left|T\,\mathrm{Re}(e^{i\theta}x)\right| \,\mathrm{d}\theta \le \frac{1}{4}\int_0^{2\pi} T\left|\mathrm{Re}(e^{i\theta}x)\right| \,\mathrm{d}\theta = T|x|,$$

where the second equality holds since T is real, and the inequality in the middle uses the established inequality for vectors in $E_{\mathbb{R}}$. □

In the study of positive operators on Banach lattices, it is mandatory to present the following delightful result.

Theorem 4.3.3 (Automatic continuity of positive operators)**.** *Let E, F be Banach lattices over the same scalar field and let $T\colon E \to F$ be a positive linear map. Then T is continuous.*

Proof. We proceed by contradiction. If T is not continuous, then we can find a sequence $(x_n) \subset E$ with $\|x_n\| = 1$ and $\|Tx_n\| \ge n^3$ all $n \in \mathbb{N}$. Moreover, replacing x_n with $|x_n|$, we may assume that $x_n \ge 0$ for all $n \in \mathbb{N}$. Observe that $x := \sum_{n=1}^\infty \frac{x_n}{n^2}$ is a well-defined element of E – as the series converges absolutely and E is complete – that satisfies $0 \le \frac{x_n}{n^2} \le x$ for all $n \in \mathbb{N}$. Thus, using the positivity of T and the lattice norm property, we find

$$\|Tx\| \ge \left\|T\left(\frac{x_n}{n^2}\right)\right\| \ge \frac{1}{n^2}\cdot n^3 = n$$

for all $n \in \mathbb{N}$, a contradiction. □

4.4 Dual spaces of Banach lattices

Given a real Banach lattice E, we introduce a partial order $\le$ on E' by defining $\varphi \le \psi$ for two functionals if $\langle \varphi, x\rangle \le \langle \psi, x\rangle$ for all $x \in E_+$. Functionals satisfying $\varphi \ge 0$ are called **positive**; these are precisely the functionals positive in the operator sense (Definition 4.3.1). The set of all positive functionals in E' is denoted by E'_+ and is called the **dual cone**.

Example 4.4.1 (The dual cone of $\mathbb{R}^n_+$). Endow $E = \mathbb{R}^n$ with the standard order and the Euclidean norm and identify E' with the space $\mathbb{R}^{1\times n}$ of row vectors.

(a) A functional $\varphi \in E'$ is positive (in the sense defined above this example) if and only if all its entries are ≥ 0 when φ is viewed as an element of $\mathbb{R}^{1\times n}$.

Hence, the positive cone E'_+ agrees with the standard cone $\mathbb{R}^{1\times n}_+$. In particular, E' is a Banach lattice.

(b) For every $\varphi \in E'$ and every $x \in E_+$ one has $\langle |\varphi|, x\rangle = \max\{|\langle \varphi, y\rangle| : y \in \mathbb{R}^n, |y| \le x\}$.

Proof. (a) This follows immediately from the characterisation of positive matrices in terms of the action in Proposition 1.1.3.

(b) Let $\varphi \in E' = \mathbb{R}^{1\times n}$ and $x \in E_+ = \mathbb{R}^n_+$. Since the order on E' is the standard order, the lattice operations on it are given componentwise (see Example 4.1.4(a)).

To obtain the claimed equality, observe that for every $y \in \mathbb{R}^n$ with $|y| \le x$ one has $\langle |\varphi|, x\rangle = |\varphi| x \ge |\varphi||y| \ge |\varphi y| = |\langle \varphi, y\rangle|$, where the second inequality uses Proposition 1.1.5(a). On the other hand, if one chooses $y \in \mathbb{R}^n$ with $y_k = x_k$ if $\varphi_k \ge 0$ and $y_k = -x_k$ if $\varphi_k < 0$, then $|y| = x$ and $\langle |\varphi|, x\rangle = \langle \varphi, y\rangle$. □

The preceding example motivates the following general result about duals of Banach lattices. For a complex Banach lattice E with real part $E_\mathbb{R}$, observe that a functional $\varphi \in E'$ is real in the sense of Definition 4.2.9 if and only $\varphi(E_\mathbb{R}) \subseteq \mathbb{R}$.

Theorem 4.4.2 (Duals of Banach lattices).

(a) *If E is a real Banach lattice, then so is E'.*

(b) *Let E be a complex Banach lattice with real part $E_\mathbb{R}$. Then E' is a complex Banach lattice whose real part consists of the real functionals in E' and can be identified with $E'_\mathbb{R}$ by restricting each such functionals to $E_\mathbb{R}$.*

In both cases one has the ***Riesz–Kantorovich formula***

$$\langle |\varphi|, x\rangle = \sup\{|\langle \varphi, y\rangle| \ : \ |y| \le x\} = \sup\{\langle \varphi, y\rangle \ : \ |y| \le x\}$$

for all $\varphi \in E'$ and $x \in E_+$, and the inequality $|\langle \varphi, z\rangle| \le \langle |\varphi|, |z|\rangle$ for all $\varphi \in E'$ and $z \in E$.

For the proof of Theorem 4.4.2 in the real case, we refer to the classical books on Banach lattices, e.g. [MN91, Theorem 1.3.2 and Proposition 1.3.7]. For the complex case see e.g. [MN91, Proposition 2.2.6] and the sentence thereafter, or [Sch74, Corollary 3 to Theorem IV.1.8].

Remark 4.4.3 (The dual cone of complex Banach lattice)**.** If E is a complex Banach lattice, then by Remark 4.2.6 and Theorem 4.4.2(b), the cone E'_+ consists of the real functionals φ whose restriction to $E_\mathbb{R}$ is positive. One readily checks that these are precisely all the $\varphi \in E'$ that are positive operators from E to $\mathbb{C}$ in the sense of Definition 4.3.1(b), so E'_+ consists precisely of the positive functionals in the complex case as well.

Finally, we discuss how positivity of operators behaves with respect to duality. To this end, we need the following consequence of the Hahn–Banach separation theorem.

Proposition 4.4.4. *Let E be a Banach lattice. An element $x \in E$ is positive if and only if $\langle \varphi, x \rangle \geq 0$ for all $\varphi \in E'_+$.*

Proof. "⇒": This implication is obvious.

"⇐": Assume that $x \notin E_+$. Since the convex set E_+ is closed (Proposition 4.1.7(c)), the Hahn-Banach separation theorem (see e.g. [Bre11, Theorem 1.7]) shows that there exists a functional $\varphi \in E'$ that strictly separates $\{x\}$ from E_+, i.e. there exists $\alpha \in \mathbb{R}$ such that $\langle \varphi, x \rangle < \alpha < \langle \varphi, y \rangle$ for all $y \in E_+$. In particular, taking $y = 0$, we get $\alpha < 0$.

We show that φ is positive. To this end, fix an arbitrary $0 \neq y \in E_+$. Then $\lambda^{-1} y \in E_+$ for all $\lambda > 0$, and thus $\lambda\alpha < \langle \varphi, y \rangle$ for all $\lambda > 0$. As $\lambda \downarrow 0$, we obtain $0 \leq \langle \varphi, y \rangle$. Thus $\varphi \in E'_+$ with $\langle \varphi, x \rangle < 0$. ☐

Corollary 4.4.5. *Let E, F be Banach lattices over the same field and $T \in \mathcal{L}(E, F)$. Then T is positive if and only if T' is positive.*

Proof. The implication "⇒" follows immediately from the definitions, and the implication "⇐" is a consequence of Proposition 4.4.4. ☐

Exercises for Chapter 4

Exercise 4.1 (Getting acquainted with lattice operations)**.**

(a) Let V be a real vector lattice and let $x, y \in V$. Derive directly from the axioms (I) and (II) in Definition 4.1.1 and from the definition of the modulus in Definition 4.1.2 that the triangle inequality $|x+y| \le |x| + |y|$ holds for all $x, y \in V$ (i.e. prove the first part of Proposition 4.1.3(e)).

(b) Consider the real Banach lattice $\mathrm{C}([0,1];\mathbb{R})$ and the functions $f_n \in \mathrm{C}([0,1];\mathbb{R})$ given by $f_n(x) = x^{1/n}$ for each $n \in \mathbb{N}$ and all $x \in [0,1]$.

Show that the set $\{f_n : n \in \mathbb{N}\}$ has a supremum f in $\mathrm{C}([0,1];\mathbb{R})$. Does the equality $f(x) = \sup_{n\in\mathbb{N}} f_n(x)$ hold for all $x \in [0,1]$?

(c) Find a sequence (g_n) in $\mathrm{C}([0,1];\mathbb{R})$ that satisfies $0 \le g_n \le \mathbb{1}$ for all n and such that the set $\{g_n : n \in \mathbb{N}\}$ does not have a supremum in the space $\mathrm{C}([0,1];\mathbb{R})$.

Exercise 4.2 (C^1 is not a vector lattice)**.** Endow the space $\mathrm{C}([-1,1];\mathbb{R})$ with the partial order $\le$ inherited from $\mathrm{C}([-1,1];\mathbb{R})$, i.e. functions are compared pointwise. Note that this space satisfies property (I) in Definition 4.1.1.

(a) Starter: Show that $\mathrm{C}^1([-1,1];\mathbb{R})$ is not a vector sublattice of $\mathrm{C}([-1,1];\mathbb{R})$.

(b) Show that $\mathrm{C}^1([-1,1];\mathbb{R})$ is not a vector lattice.

Exercise 4.3 (Positivity of the resolvent for Δ_{Dir})**.** Let $\emptyset \ne \Omega \subseteq \mathbb{R}^n$ be open and consider the Dirichlet Laplacian $\Delta_{\mathrm{Dir}} \colon L^2(\Omega) \supseteq \operatorname{dom}(\Delta_{\mathrm{Dir}}) \to L^2(\Omega)$ from Example 3.3.6 given by

$$\operatorname{dom}(\Delta_{\mathrm{Dir}}) := \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}, \qquad \Delta_{\mathrm{Dir}} u := \Delta u.$$

Let $\lambda \in (0,\infty)$. According to Example 3.3.6(c) one has $\lambda \in \rho(\Delta_{\mathrm{Dir}})$. In this exercise we show that the resolvent $\mathcal{R}(\lambda, \Delta_{\mathrm{Dir}}) \colon L^2(\Omega) \to L^2(\Omega)$ is a positive operator.

(a) Let $0 \le g \in L^2(\Omega)$ and set $u := \mathcal{R}(\lambda, \Delta_{\mathrm{Dir}})g \in \operatorname{dom}(\Delta_{\mathrm{Dir}})$. Derive from the description of Δ_{Dir} in Example 3.3.6(b) that

$$\lambda (v \mid u)_{L^2} + (\nabla v \mid \nabla u)_{L^2} = (v \mid g)_{L^2} \tag{4.4.1}$$

for all $v \in H_0^1(\Omega)$. Are we allowed to substitute $v := u^-$ in this equation?

(b) Show that $u^- = 0$ and conclude that $\mathcal{R}(\lambda, \Delta_{\mathrm{Dir}}) \geq 0$.

Exercise 4.4 (Range of a positive projection)**.**

(a) Let V be a real vector lattice and $P \in \mathcal{L}(V)$ a positive projection.[6] Show that $\operatorname{rg} P$ is a vector lattice with respect to the order inherited from V and the corresponding cone $(\operatorname{rg} P)_+ := \{v \in \operatorname{rg} P : v \geq 0\}$ satisfies $(\operatorname{rg} P)_+ = \operatorname{rg} P \cap V_+ = P(V_+)$.

Give an explicit formula for the modulus in $\operatorname{rg} P$. Is $\operatorname{rg} P$ always a vector sublattice?

(b) Let E be a real Banach lattice and $P \in \mathcal{L}(E)$ a positive projection. Show that $\|x\|_{\operatorname{rg} P} := \|P|x|\|$ for all $x \in E$ defines an equivalent norm on $\operatorname{rg} P$ with respect to which $\operatorname{rg} P$ is a Banach lattice.

(c) Let E be a complex Banach lattice and $P \in \mathcal{L}(E)$ a positive projection. Observe that $\operatorname{rg} P$ is the vector space complexification of the Banach lattice $P(E_{\mathbb{R}})$ and thus becomes a complex Banach lattice by Definition 4.2.5.

Give an explicit formula for the complex modulus function $\operatorname{rg} P \to P(E_{\mathbb{R}})$ and for the norm that turns $\operatorname{rg} P$ into a complex Banach lattice.

(d) Let $A \subseteq \mathrm{C}([0,1];\mathbb{R})$ denote the two-dimensional subspace consisting of all affine functions endowed with the norm and the order inherited from $\mathrm{C}([0,1];\mathbb{R})$.

Show that A is not a vector sublattice of $\mathrm{C}([0,1];\mathbb{R})$ but is itself a Banach lattice.

Exercise 4.5. In this exercise, we will use some general notions from order theory. A **directed set** is a non-empty set J together with a reflexive and transitive relation $\leq$ such that for all $a, b \in J$, there exists $c \in J$ such that $a \leq c$ and $b \leq c$. A **net** in a set X is a function $x\colon J \to X$, where J is a directed set; it is customary to write $(x_j)_{j \in J}$ instead of x.[7]

Let E be a complex Banach lattice and let $(T_j)_{j\in J}$ be a net in the space $\mathcal{L}(E)$ of bounded linear operators on E. Assume that $(T_j)_{j\in J}$ is an **individually eventually positive net** in the sense that for each $f \in E_+$, there exists $j_0 \in J$ such that $T_j f \geq 0$ for all $j \geq j_0$.

(a) Show that for each $f \in E_{\mathbb{R}}$, there exists $j_1 \in J$ such that $|T_j f| \leq T_j |f|$ for all $j \geq j_1$.

(b) Assume now that there exists a countable set $I \subseteq J$ that is **majorising**, meaning that for each $j \in J$ there exists a $j_1 \in I$ that satisfies $j_1 \geq j$. Show that $(T_j)_{j\in J}$ is **uniformly eventually real** in the sense that there exists $j_0 \in J$ such that T_j is a real operator for all $j \in J$ with $j \geq j_0$.

Hint: Use Baire's theorem.

[6] Recall that a linear map is called a **projection** if it satisfies $P^2 = P$.
[7] Note that every sequence is a net with $J = \mathbb{N}$, endowed with its usual order.

Notes for Chapter 4

Vector lattices, Banach lattices, and ordered vector spaces

Standard literature

Vector lattices are also called **Riesz spaces** in the literature. The theory dates back to the first half of the twentieth century, but we refrain from trying to give a historical account of its development. Standard books on the topic include: the two volumes by Luxemburg and Zaanen [LZ71] and Zaanen [Zaa83] – the first of which focuses on vector lattices without norms while the latter has more material about Banach lattices; Schaefer's classical monograph [Sch74] on Banach lattices; the reprint [AB06] of a classical book by Aliprantis and Burkinshaw from the '80s; the book of Meyer–Nieberg [MN91], which might sometimes be a bit less heavy on notation and technicalities; and more recently, the introductory textbook [Zaa97] by Zaanen. We should also mention the book [Wnu99] by Wnuk which gives a comprehensive treatment of the important class of Banach lattices with **order continuous norm**.

Terminological remarks

As Section 4.1 shows, the theory of vector lattices is most naturally developed over the real field. It is thus customary in the literature to simply call "vector lattice" and "Banach lattice" what we call a "real vector lattice" and "real Banach lattice", respectively. Since we use the complex scalar field frequently and also discuss a number of results that are true over both scalar fields, we find it more natural to use the term "Banach lattice" for both cases simultaneously and specify the field whenever necessary.

Ordered vector spaces and ordered Banach spaces

Many classes of vector spaces and Banach spaces carry a canonical structure that does not turn them into vector lattices. Typical examples are spaces of differentiable functions like $C^k(\Omega)$ (see for instance Exercise 4.2), higher order Sobolev spaces [AN09, Example 2.3(d)], and the self-adjoint parts of non-commutative C^*-algebras; the latter are never lattice ordered by a classical result of Sherman [She51, Theorems 1 and 2]. Still, these spaces are often Banach spaces. Important results on such ordered Banach spaces can, for instance, be found in the first part of the classical paper [BR84] by Batty and

Robinson, in the book [KLS89] by Krasnosel'skii, Lifshits, and Sobolev, and [AT07, Section 2.5] of an excellent book by Aliprantis and Tourky. More recently, the German translation [Wul17] by Weber of two classical Russian books by Wulich appeared and contains some notes on more recent development of the field.

A very recent topic is the study of ordered vector spaces by embedding them into vector lattices – this is the theory of **pre-Riesz spaces** which is presented in the book [KvG19] by Kalauch and van Gaans.

As pointed out in Example 4.1.4(d), the Stampacchia lemma implies that the first order Sobolev spaces $W^{1,p}(\Omega;\mathbb{R})$ are vector lattices. However, they are not Banach lattices according to Example 4.1.6(d), and the same argument – considering quickly oscillating functions – shows that they do not become Banach lattices with respect to any equivalent norm. However, they are vector lattices and Banach spaces at the same time and their positive cone is closed. It seems that, from an abstract point of view, such "lattice-ordered Banach spaces" have not been studied systematically in the literature, but some interesting results about them were proved by Borwein and Yost in [BY84].

Complexification

Existence of a complex modulus function

Complexifications of Banach lattices go back to Lotz who developed them to study the spectrum of positive operators [Lot68]. His approach was based on the formula (4.2.1) for complex numbers: For an element z of the vector space complexification $E_{\mathbb{C}}$ of a real Banach space E he defined the complex modulus as

$$|z|_{\mathbb{C}} := \sup_{\theta\in[0,2\pi]} \operatorname{Re}(e^{i\theta} z) \tag{4.4.2}$$

and then showed that it has the properties (I)–(III) stipulated in Definition 4.2.3. The approach still seems to be the most common one in the literature. Its main obstacle is that it is not at all clear why this supremum exists – note that the definition of a vector lattice only requires the existence of the supremum of sets of two elements (and hence of finite sets). The most common way to solve this problem is to use a representation result due to Kakutani about a class of Banach lattices, called **AM-spaces**. This result enables reducing the situation to spaces of continuous functions where one can explicitly check that the supremum exists. We refer for instance to [Sch74, Section II.11] for the details of this procedure. A different approach is used in [Zaa97, Chapter 6] where the existence of the supremum is proved intrinsically – i.e. without employing a representation result – by using a certain type of convergence that is often called **relative uniform convergence**.

By contrast, our existence proof for a complex modulus function relies on the integral formula (4.2.2). To the best of our knowledge, this formula has not been used in the construction of complex vector lattices so far – which is remarkable since it actually gives existence under more general assumptions than the more common procedures which require the vector lattice to be **relatively uniformly complete**.[8]

While formula (4.2.2) does not seem to have found its way into the standard literature on Banach lattices, representing the modulus in this way is known to be a useful technique in related parts of functional analysis. For instance, consider a bounded linear operator T on a real-valued L^p-space $L^p(\Omega,\mu;\mathbb{R})$ and let $T_\mathbb{C}$ denote its canonical extension to the complex-valued space $L^p(\Omega,\mu)$. Then we can prove that $\|T\| = \|T_\mathbb{C}\|$. This is not obvious at all, but it can be proved by using an L^p-version of the integral representation of the complex modulus, see e.g. [Fen98, Proposition 2.1.1 and Remark 2.1.1]

Uniqueness of the complex modulus function

Uniqueness of the complex modulus function was proved by Mittelmeyer and Wolff in [MW74, Theorem 2.2]. It is valid under much weaker assumptions than existence results – one only needs the underlying real vector lattice to be **Archimedean**, a property that is typically satisfied in non-pathological examples. Remarkably, the theory of complex Banach lattices can essentially be developed without discussing uniqueness. Indeed, the standard books on Banach lattices define the complex modulus by the formula in (4.4.2) (or a version thereof) and then derive the property that one would expect from a modulus function without ever discussing uniqueness. From this perspective, one could analogously define the theory of complex Banach lattices by defining the complex modulus in terms of the integral formula in Theorem 4.2.4 without bothering about uniqueness.

Apart from purely theoretical interest, the one point where knowledge of the uniqueness of the complex modulus function is quite handy, is to check that concrete examples of complex functions spaces are indeed complex Banach lattices. The efficiency of this approach is demonstrated in Examples 4.2.8. Without the uniqueness result, showing that the common function spaces are indeed complex Banach lattices requires a bit more work. For L^p-spaces one can for instance do this by showing that it suffices to takes the supremum in $\sup_{\theta\in[0,2\pi]}\operatorname{Re}(e^{i\theta}z)$ over a countable subset of $[0,2\pi]$ and then by working pointwise almost everywhere. Alternatively, if one takes our integral formula approach, it can be shown that L^p-spaces are complex Banach lattices by using results about the almost-everywhere evaluation of L^p-valued Bochner integrals (see e.g. [HvNVW16, Proposition 1.2.25 and Remark 1.2.26]).

More on complexifications

The norm on a complex Banach lattice is closely related to tensor products of Banach spaces as shown in [vN97].

Complexifications can be defined not only for Banach lattices, but also make sense for real Banach spaces. In this case one does, of course, not define a complex modulus function, but focuses on ways to extend the norm on a real Banach space to its vector space complexification. We refer to the paper [MST99] for an overview.

[8] Note that we have not presented the results in the main text under optimal assumptions by any means, to keep the presentation concise and accessible and in order to focus on the key ideas.

Finally, we point out that our definition of the complexification of real vector spaces (Definition 4.2.1) is rather explicit and not axiomatic. This makes the definition itself rather straightforward to digest, but it has the disadvantage that it requires certain identifications later on when wants to consider a vector space that is not explicitly given as $V \times V$ as the complexification of the real vector space V. Such identifications already occur for instance in Examples 4.2.8 and they become more explicit in Exercise 4.4(c) and, in particular, in Theorem 4.4.2(b). For a presentation of complexifications that is more axiomatic and thus makes all identifications explicit, we refer to [Glü16, Appendix C].

The Riesz–Kantorovich formula(s)

The Riesz–Kantorovich formula from Theorem 4.4.2 also holds more generally for so-called **regular** operators between a Banach lattice E and a **Dedekind complete** Banach lattice F. For the case of real scalars, see e.g. [AB06, Theorem 1.18]; there, one can also find similar formulas for the positive and the negative part of a regular operator. For the complex case, we refer to [Sch74, Definition IV.1.7 and Theorem IV.1.8].

Appendices

4.A Vector-valued integrals

In this appendix, we present some essential aspects of integration theory for functions with values in a Banach space. A good overview of elementary concepts can be found in [AE01, Kapitel X.1 & X.2], and [HvNVW16, Chapter 1] offers a comprehensive introduction. The stochastically-minded reader can also consult [DPZ14, Section 1.1].

To keep things simple, we only consider σ-finite measure spaces throughout this section, although much of the theory can also be developed in more generality. Recall that a measure space $(\Omega, \mathscr{A}, \mu)$ is called **σ-finite** if there exist countably many measurable sets $A_n \in \mathscr{A}$ with $\mu(A_n) < \infty$ and $\bigcup_{n\in\mathbb{N}} A_n = \Omega$.

Measurability

As a first reaction, one might think: "what's the big deal?" If $(\Omega, \mathscr{A})$ is a measurable space and X is a Banach space, then surely it makes sense to say that a function $f\colon \Omega \to X$ is measurable if and only if $f^{-1}(B) \in \mathscr{A}$ for every Borel subset B in X. It turns out that this natural definition is not as useful as it looks. Often in functional analysis, the norm dual X' is used to study problems on X by reducing to the case of scalar-valued functions. This is supported by the following simple result.

Proposition 4.A.1. *Let X be a separable Banach space, and let $\mathscr{A}$ be the σ-algebra generated by all sets of the form*

$$\{x \in X : \langle x', x\rangle \le \alpha\}, \quad x' \in X', \alpha \in \mathbb{R}. \tag{4.A.1}$$

Then $\mathscr{A}$ coincides with the Borel σ-algebra $\mathscr{B}(X)$.

Proof. The inclusion $\mathscr{A} \subseteq \mathscr{B}(X)$ is immediate, since each set of the form (4.A.1) is closed and hence in $\mathscr{B}(X)$. To prove the converse, it suffices to show that $\mathscr{A}$ contains all open balls in X. Since X is separable, there is a countable norming sequence $(x_n') \subset X'$, i.e.

$$\|x\| = \sup_{n\in\mathbb{N}} \left|\langle x_n', x\rangle\right| \quad \forall\, x \in X.$$

Consequently, for arbitrary $x \in X$ and $r > 0$, it holds that

$$B(x; r) = \bigcup_{m=1}^{\infty} \overline{B}\left(x; r\left(1 - \tfrac{1}{m}\right)\right) = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \{y \in X : \left|\langle x_n', y - x\rangle\right| \le r\left(1 - \tfrac{1}{m}\right)\} \in \mathscr{A}. \qquad \square$$

However, if the Banach space X is not separable, then it may happen that the σ-algebra generated by X' is strictly smaller than the Borel σ-algebra $\mathscr{B}(X)$ (so in some sense $\mathscr{B}(X)$ is 'too large'). In this case, the measurability properties on X cannot be effectively determined via linear functionals.

On the other hand, we often study measurability and integrability of scalar-valued functions via approximation by simple functions. These considerations motivate the following.

Definition 4.A.2. Let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space and X a Banach space.

(a) A function $\varphi \colon \Omega \to X$ is called **μ-simple** if it satisfies: (I) $\varphi(\Omega)$ is a finite subset of X; (II) $\varphi^{-1}(\{x\}) \in \mathscr{A}$ for all $x \in X$; and (III) $\mu\big(\varphi^{-1}(X \setminus \{0\})\big) < \infty$.

(b) A function $f \colon \Omega \to X$ is called **strongly μ-measurable** if there is a sequence $(f_n)_{n\in\mathbb{N}}$ of μ-simple functions such that $\lim_{n\in\mathbb{N}} f_n(\omega) = f(\omega)$ for μ-almost every $\omega \in \Omega$.

In the case $X = \mathbb{C}$, it is common to omit the word 'strongly' and simply speak of μ-measurable functions $f \colon \Omega \to \mathbb{C}$.

(c) A function $f \colon \Omega \to X$ is called **weakly μ-measurable** if the $\mathbb{C}$-valued map $x' \circ f = \langle x', f(\cdot)\rangle$ is μ-measurable for all $x' \in X'$.

Remark 4.A.3. Let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space and X a Banach space. Every μ-simple function $\varphi \colon \Omega \to X$ can be written uniquely in the form

$$\varphi = \sum_{k=1}^{N} \mathbb{1}_{A_k} x_k \tag{4.A.2}$$

with distinct elements $\{x_1, \dots, x_N\} \subset X$, and where the sets $A_k \in \mathscr{A}$ satisfy $\mu(A_k) < \infty$ for each $k = 1, \dots, N$ and are pairwise disjoint, i.e. $A_j \cap A_k = \emptyset$ whenever $j \neq k$. Indeed, let $\{x_1, \dots, x_N\} \subset X$ be the disinct non-zero values of φ, and set $A_k := \varphi^{-1}(\{x_k\})$. Clearly the A_k's are pairwise disjoint and $\mu(A_k) < \infty$ for each $k = 1, \dots, N$. The uniqueness of the representation (4.A.2) can then be easily verified.

Definition 4.A.2 does not require the Banach space X to be separable. This is important, since non-separable Banach spaces (such as $L^\infty(\Omega)$ for an open subset $\emptyset \neq \Omega \subseteq \mathbb{R}^n$) are also useful and appear in many applications. Nevertheless, Proposition 4.A.1 gives a hint that if we want a good integration theory, separability should not be far away.

Definition 4.A.4. Let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space and X a Banach space. A function $f \colon \Omega \to X$ is called **μ-almost separably valued** if there exists a separable subset[9] X_0 of X such that $f(\omega) \in X_0$ for μ-almost every $\omega \in \Omega$.

We can now formulate the fundamental theorem of Pettis.

[9]Naturally, we equip X_0 with the relative topology inherited from X. Note that some texts require X_0 to be a closed, separable vector subspace of X. However, it can be shown that the two definitions are equivalent.

Theorem 4.A.5 (Pettis)**.** *Let* $(\Omega, \mathscr{A}, \mu)$ *be a* σ*-finite measure space and* X *a Banach space. For a function* $f\colon \Omega \to X$*, the following assertions are equivalent:*

(i) f *is strongly* μ*-measurable;*

(ii) f *is* μ*-almost separably valued and weakly* μ*-measurable.*

For the proof, we refer to [HvNVW16, Theorem 1.1.20].

The Bochner integral

For strongly μ-measurable functions with values in a Banach space, one can define an integral as follows.

Definition 4.A.6 (The Bochner integral)**.** Let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space and X a Banach space.

(a) For a μ-simple function $\varphi\colon \Omega \to X$, $\varphi = \sum_{k=1}^{N} \mathbb{1}_{A_k} x_k$, we define

$$\int_\Omega \varphi \,\mathrm{d}\mu := \sum_{k=1}^{N} \mu(A_k) x_k. \tag{4.A.3}$$

(b) Let $f\colon \Omega \to X$ be a strongly μ-measurable function. We say that f is **Bochner integrable** (with respect to μ) if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of μ-simple functions such that

$$\lim_{n\to\infty} \int_\Omega \|f_n - f\| \,\mathrm{d}\mu = 0.$$

In this case, we define

$$\int_\Omega f \,\mathrm{d}\mu := \lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu. \tag{4.A.4}$$

Remarks 4.A.7.

(a) By the uniqueness assertion in Remark 4.A.3, the integral for simple functions (4.A.3) is well-defined. Moreover, it is clear that $\left\|\int_\Omega \varphi \,\mathrm{d}\mu\right\| \le \int_\Omega \|\varphi\| \,\mathrm{d}\mu$, and it is straightforward to verify that additivity $\int_\Omega \varphi \,\mathrm{d}\mu + \int_\Omega \psi \,\mathrm{d}\mu = \int_\Omega \varphi + \psi \,\mathrm{d}\mu$ holds for all simple functions $\varphi, \psi\colon \Omega \to X$ (cf. [AE01, Kapitel X.2, Bemerkungen 2.1] for details).

(b) To see that (4.A.4) is well-defined, first note that the map $\Omega \ni \omega \mapsto \|f_n(\omega) - f(\omega)\| \in [0,\infty)$ is μ-measurable, since it is a composition of the strongly μ-measurable function $f_n - f$ with the continuous function $\|\cdot\|\colon X \to [0,\infty)$. By the properties of the integral for simple functions mentioned above, one has

$$\left\|\int_\Omega f_n \,\mathrm{d}\mu - \int_\Omega f_m \,\mathrm{d}\mu\right\| \le \int_\Omega \|f_n - f_m\| \,\mathrm{d}\mu \le \int_\Omega \|f_n - f\| \,\mathrm{d}\mu + \int_\Omega \|f_m - f\| \,\mathrm{d}\mu.$$

Hence if f is Bochner integrable, then the integrals $\int_\Omega f_n \,\mathrm{d}\mu$ form a Cauchy sequence in X, and thus converge to a unique element.

Theorem 4.A.8 (Bochner)**.** *Let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space and X a Banach space. A strongly μ-measurable function $f\colon \Omega \to X$ is Bochner integrable if and only if*

$$\int_\Omega \|f\| \, \mathrm{d}\mu < \infty.$$

In this case, it holds that

$$\left\| \int_\Omega f \, \mathrm{d}\mu \right\| \le \int_\Omega \|f\| \, \mathrm{d}\mu.$$

We refer to [HvNVW16, Proposition 1.2.2] for the proof. In the case that the measure space is an interval $I \subseteq \mathbb{R}$ with the Lebesgue measure, proofs of Theorems 4.A.5 and 4.A.8 can also be found in [ABHN11, Theorem 1.1.1] and [ABHN11, 1.1.4] respectively.

Example 4.A.9. Let λ^n be the Lebesgue measure on $\mathbb{R}^n$, $K \subset \mathbb{R}^n$ a non-empty compact subset, and X a Banach space. Then every continuous function $f\colon K \to X$ is strongly λ^n-measurable. Indeed, the image $f(K)$ is a compact subset of X, and hence separable, so in particular f is λ^n-almost separably valued. (More generally, if $\Omega \subseteq \mathbb{R}^n$ can be covered by countably many compact sets, then every continuous function $f\colon \Omega \to X$ is separably valued). For each $x' \in X'$, the $\mathbb{C}$-valued map $x' \circ f$ is continuous (as a composition of continuous maps) and hence λ^n-measurable. The conclusion now follows from Theorem 4.A.5.

Moreover, Theorem 4.A.8 shows that every continuous function $f\colon K \to X$ is also Bochner integrable, since one has

$$\int_K \|f(\omega)\| \, \mathrm{d}\lambda^n \le \sup_{\omega \in K} \|f(\omega)\| \cdot \lambda^n(K) < \infty.$$

Finally, we present a useful result that explains how closed linear operators interact with Bochner integrals.

Theorem 4.A.10 (Hille)**.** *Let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space and X, Y Banach spaces. Suppose that $f\colon \Omega \to X$ is Bochner integrable, and $A\colon X \supseteq \operatorname{dom}(A) \to Y$ is a closed linear operator. Assume that $f(\omega) \in \operatorname{dom}(A)$ for μ-almost every $\omega \in \Omega$ and that $A \circ f\colon \Omega \to Y$ (defined for μ-almost every $\omega \in \Omega$) is Bochner integrable. Then f is Bochner integrable as a $\operatorname{dom}(A)$-valued function, $\int_\Omega f \, \mathrm{d}\mu \in \operatorname{dom}(A)$, and*

$$A \int_\Omega f \, \mathrm{d}\mu = \int_\Omega A \circ f \, \mathrm{d}\mu. \tag{4.A.5}$$

We refer to [HvNVW16, Theorem 1.2.4] for the proof. Observe that if $A \in \mathcal{L}(X, Y)$, then (4.A.5) can be derived easily from Definition 4.A.6 alone.

Encore: If you want to know more. . .

4.B The lattice structure of $W^{1,p}$

In Example 4.1.4(d), it was mentioned that the Sobolev spaces $W^{1,p}(\Omega)$ are vector lattices. In this appendix, we give a proof of this fact, which is a consequence of Theorem 4.B.3 due to Stampacchia. This is not merely a curiosity for vector lattice theory, but it turns out to be a very useful property for PDE theory in general.

As preparation, we require a Sobolev space version of the chain rule (which is also a very useful tool in itself).

Proposition 4.B.1 (A chain rule for $W^{1,p}$)**.** *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open, and $u \in W^{1,p}(\Omega)$ for some $p \in [1,\infty]$. Let $G \in \mathrm{C}^1(\mathbb{R};\mathbb{R})$ satisfy $G' \in L^\infty(\mathbb{R})$ and $G(0) = 0$. Then the composition $G \circ u$ belongs to $W^{1,p}(\Omega)$ and*

$$\partial_k(G \circ u) = (G' \circ u)\partial_k u, \qquad k = 1,\dots,d. \tag{4.B.1}$$

Proof. The assumptions on G and the mean value theorem imply that $|G(x)| \leq M|x|$ for all $x \in \mathbb{R}$, with $M = \|G'\|_{L^\infty(\mathbb{R})}$. Hence $|G \circ u| \leq M|u|$, which shows that $G \circ u \in L^p(\Omega)$. Since $G' \in L^\infty(\mathbb{R})$, we also have $(G' \circ u)\partial_k u \in L^p(\Omega)$ for all $k \in \{1,\dots,n\}$.

We treat the case $p < \infty$ first; this allows us to leverage the Meyers-Serrin theorem (Theorem 3.A.4), and choose a sequence $(u_n) \in \mathrm{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$. Hence, by the classical integration by parts and chain rule, we obtain

$$\int_\Omega (G \circ u_n)\partial_k \varphi \, \mathrm{d}x = -\int_\Omega (G' \circ u_n)(\partial_k u_n)\varphi \, \mathrm{d}x \tag{4.B.2}$$

for all $n \in \mathbb{N}$ and all $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$. The inequality $|G \circ u_n - G \circ u| \leq M|u_n - u|$ implies that $G \circ u_n \to G \circ u$ in $L^p(\Omega)$. Furthermore, since G' is continuous, $G' \circ u_n$ converges pointwise a.e. to $G' \circ u$, and hence $(G' \circ u_n)\partial_k u_n \to (G' \circ u)\partial_k u$ in $L^p(\Omega)$ by dominated convergence. By letting $n \to \infty$ in (4.B.2), we therefore obtain

$$\int_\Omega (G \circ u)\partial_k \varphi \, \mathrm{d}x = -\int_\Omega (G' \circ u)(\partial_k u)\varphi \, \mathrm{d}x$$

for all $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$. This proves (4.B.1) in the case $p \neq \infty$.

The case $p = \infty$ is handled via a simple trick. Given $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$, choose a bounded subset Ω' such that $\operatorname{supp}\varphi \subseteq \Omega'$ and $\overline{\Omega'} \subseteq \Omega$. Then it follows that $u \in W^{1,p}(\Omega')$ for any $1 \leq p < \infty$, and the previous arguments can be applied. □

Remark 4.B.2. The assumption that $G(0) = 0$ in Proposition 4.B.1 is only needed to obtain $G \circ u \in L^p(\Omega)$ (as the proof readily reveals). If one is not concerned about p-integrability, this assumption on G can be dropped; cf. [GT01, Lemma 7.5].

Theorem 4.B.3 (Stampacchia)**.** *Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be open and $p \in [1,\infty]$. If $u \in W^{1,p}(\Omega)$, then $u^+, u^-, |u| \in W^{1,p}(\Omega)$ with*

$$\partial_k(u^+) = \begin{cases} \partial_k u & \text{on } u > 0 \\ 0 & \text{on } u \leq 0, \end{cases} \qquad \partial_k(u^-) = \begin{cases} 0 & \text{on } u \geq 0 \\ -\partial_k u & \text{on } u < 0, \end{cases} \qquad \partial_k |u| = \begin{cases} \partial_k u & \text{on } u > 0 \\ 0 & \text{on } u = 0 \\ -\partial_k u & \text{on } u < 0, \end{cases}$$

for all $k \in \{1, \dots, d\}$. In particular, $W^{1,p}(\Omega)$ is a vector sublattice of $L^p(\Omega)$.

Proof. For every $\varepsilon > 0$, we define a smooth approximation of the function $x \mapsto x^+$ by

$$G_\varepsilon(x) := \begin{cases} (x^2 + \varepsilon^2)^{1/2} - \varepsilon & x > 0 \\ 0 & x \leq 0. \end{cases}$$

By Proposition 4.B.1, it holds that

$$\int_\Omega (G_\varepsilon \circ u)\partial_k \varphi \, \mathrm{d}x = -\int_\Omega \frac{u \partial_k u}{(u^2 + \varepsilon^2)^{1/2}} \varphi \, \mathrm{d}x = -\int_{\{u>0\}} \frac{u \partial_k u}{(u^2 + \varepsilon^2)^{1/2}} \varphi \, \mathrm{d}x$$

for all $\varphi \in C_c^\infty(\Omega)$. Since $G_\varepsilon \circ u \to u^+$ pointwise a.e. in Ω and $u(u^2 + \varepsilon^2)^{-1/2} \to 1$ pointwise a.e. in the subset $\{u > 0\}$ as $\varepsilon \downarrow 0$, the above equalities in the limit yield

$$\int_\Omega u^+ \partial_k \varphi \, \mathrm{d}x = -\int_{\{u>0\}} (\partial_k u)\varphi \, \mathrm{d}x.$$

This proves the required formula for $\partial_k(u^+)$.

The formulae for $\partial_k(u^-)$ and $\partial_k |u|$ now follow immediately from the identities $u^- = (-u)^+$ and $|u| = u^+ + u^-$. The lattice operations of supremum and infimum, naturally inherited from $L^p(\Omega)$, can then be expressed in terms of the modulus, as shown in Proposition 4.1.3(f). This proves that $W^{1,p}(\Omega)$ is a vector sublattice of $L^p(\Omega)$. □

Remark 4.B.4. One can show that the lattice operations $u \mapsto u^+$, $u \mapsto u^-$ and $u \mapsto |u|$ are continuous on $W^{1,p}(\Omega)$ (this does not follow from Proposition 4.1.7, which only yields continuity on $L^p(\Omega)$). The curious reader may consult, for instance, [AU23, Theorem 6.37], for the proof when $p = 2$ that can be easily adapted for general values of p.

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