Chapter 3

Unbounded operators and their spectra

In Chapters 1 and 2 we studied (eventual) positivity properties for matrix semigroups $(e^{tA})_{t\geq 0}$ and resolvents $\mathcal{R}(\lambda, A) = (\lambda - A)^{-1}$ of matrices. These objects can be used to solve different types of equations in $\mathbb{R}^n$. For $x_0 \in \mathbb{C}^n$, the mapping $t \mapsto e^{tA}x_0$ solves a linear differential equation (Corollary 1.3.3) and the vector $u := \mathcal{R}(\lambda, A)x_0$ solves the linear equation $(\lambda - A)u = x_0$ in $\mathbb{C}^n$.

From now on and throughout the rest of the course, we study the analogous equations in infinite dimensions. The equations of interest are typically partial differential equations, so the matrix A will be replaced by a differential operator on a Banach space. As we will see, these are typically unbounded operators, and hence it is the purpose of the current chapter to give an introduction to the theory of unbounded operators.

3.1 Unbounded operators

Differential operators are operators that map every function f from a suitable function space to a new function that involves the (partial) derivatives of f. Such operators cannot be defined everywhere on classical function spaces such as $\mathrm{C}([0,1])$, because not every continuous function has a derivative. This motivates the definition of linear operators that are only defined on a vector subspace of a given Banach space.

Definition 3.1.1 (Linear operators)**.** Let X, Y be Banach spaces over the same scalar field.

(a) A **linear operator**, or briefly, an **operator**, between X and Y is a linear mapping $A\colon \operatorname{dom}(A) \to Y$, where $\operatorname{dom}(A)$ is a vector subspace of X. We briefly write $A\colon X \supseteq \operatorname{dom}(A) \to Y$ for such an operator. If $X = Y$, we say that A is an operator *on* X. The space $\operatorname{dom}(A)$ is called the **domain of** A.

Now, let $A\colon X \supseteq \operatorname{dom}(A) \to Y$ be a linear operator.

(b) The operator A is said to be **everywhere defined** if $\operatorname{dom}(A) = X$. It is said to be **densely defined** or to have a **dense domain** if $\operatorname{dom}(A)$ is dense in X.

(c) The operator A is called **closed** if its graph $\{(x, Ax) : x \in \operatorname{dom}(A)\}$ is closed in $X \times Y$.

(d) A norm on $\operatorname{dom}(A)$ is called a **graph norm** of A if it is equivalent to the norm

$$\|\cdot\|_A : \operatorname{dom}(A) \to [0,\infty), \qquad x \mapsto \|x\|_X + \|Ax\|_Y .$$

When the domain $\operatorname{dom}(A)$ of an operator is endowed with a graph norm, then the inclusion map $\operatorname{dom}(A) \hookrightarrow X$ is obviously continuous. Note that a linear operator $A\colon X \supseteq \operatorname{dom}(A) \to Y$ is, in general, not be a continuous map from $\operatorname{dom}(A)$ to X if $\operatorname{dom}(A)$ is endowed with the norm induced by X; hence, one often refers to such operators as **unbounded operators**. However, A is clearly continuous when $\operatorname{dom}(A)$ is endowed with a graph norm of A. If one wants to apply the theory of bounded linear operators between Banach spaces to A, ideally $\operatorname{dom}(A)$ would be a Banach space with respect to some (hence, every) graph norm of A. We now prove that this is the case if and only if A is closed.

Proposition 3.1.2 (Characterisation of closedness)**.** *Let X, Y be Banach spaces and let $A\colon X \supseteq \operatorname{dom}(A) \to Y$ be a linear operator. The following are equivalent:*

(i) *The operator A is closed.*

(ii) *If a sequence (x_k) in $\operatorname{dom}(A)$ converges (with respect to the X-norm) to a point $x \in X$ and (Ax_k) converges to a point $y \in Y$, then $x \in \operatorname{dom}(A)$ and $Ax = y$.*

(iii) *The domain $\operatorname{dom}(A)$ is complete (hence, a Banach space) with respect to some (equivalently, every) graph norm.*

Proof. "(i) $\Leftrightarrow$ (ii)": This follows directly from the definition of closed operators.

"(ii) $\Rightarrow$ (iii)": Let (x_k) be a Cauchy sequence in $\operatorname{dom}(A)$ with respect to $\|\cdot\|_A$. Then both (x_k) and (Ax_k) are Cauchy in X, so there exist $x \in X$, $y \in Y$ such that $(x_k) \to x$ in X and $(Ax_k) \to y$ in Y. By (ii), $x \in \operatorname{dom}(A)$ and $Ax = y$, and thus

$$\|x_k - x\|_A = \|x_k - x\|_X + \|A(x_k - x)\|_Y = \|x_k - x\|_X + \left\|Ax_k - y\right\|_Y \to 0.$$

"(iii) $\Rightarrow$ (ii)": Assume (iii) and let (x_k), x, y be as in (ii). Then (x_k) and (Ax_k) are Cauchy in X and Y respectively. Hence, (x_k) is Cauchy with respect to $\|\cdot\|_A$ and thus converges to a point $w \in \operatorname{dom}(A)$ with respect to $\|\cdot\|_A$. In particular, one also has $x_k \to w$ with respect to $\|\cdot\|_X$, so $w = x$. Hence, $x \in \operatorname{dom}(A)$.

On the other hand, the convergence of $x_k \to x$ with respect to $\|\cdot\|_A$ also implies that $Ax_k \to Ax$ and therefore $Ax = y$. □

We briefly recall one of the fundamental results in functional analysis, the closed graph theorem. In the terminology from Definition 3.1.1, it can be phrased as follows.

Theorem 3.1.3 (Closed graph theorem)**.** *Let X, Y be Banach spaces and consider a linear operator $A : X \supseteq \operatorname{dom}(A) \to Y$. If A is closed and everywhere defined, then A is continuous.*

A simple but illustrative class of closed operators that are not everywhere defined are operators that act by multiplication with an unbounded function. These are easy to work with and thus useful to get a first intuition for many concepts in operator theory. You will investigate this further in Exercise 3.2. Our actual objects of interest, though, are differential operators. Let us start with two simple one-dimensional examples.

Examples 3.1.4 (Differential operators on an interval)**.**

(a) Let $A\colon \mathrm{C}([-1,1]) \supseteq \operatorname{dom}(A) := \mathrm{C}^1([-1,1]) \to \mathrm{C}([-1,1])$ be given by $Af := f'$ for all $f \in \operatorname{dom}(A)$. Then A is densely defined and closed.

(b) Let $p \in [1,\infty)$ and let $A\colon L^p(-1,1) \supseteq \operatorname{dom}(A) := \mathrm{C}^1([-1,1]) \to L^p(-1,1)$ be given by $Af := f'$ for all $f \in \operatorname{dom}(A)$. Then A is densely defined, but not closed.

Proof. (a) It follows from the Weierstraß approximation theorem that $\mathrm{C}^1([-1,1])$ is dense in $\mathrm{C}([-1,1])$, so A is densely defined. To show closedness, let (f_k) be a sequence in $\mathrm{C}^1([-1,1])$ that converges uniformly to $f \in \mathrm{C}([-1,1])$ and assume that the derivatives f_k' converges uniformly to some $g \in \mathrm{C}([-1,1])$. For every $x \in [-1,1]$ it follows that

$$f(x) = f(0) + \lim_{n\to\infty} f_k(x) = \lim_{n\to\infty} \int_0^x f_k'(y)\,\mathrm{d}y = f(0) + \int_0^x g(y)\,\mathrm{d}y.$$

Thus, $f \in \mathrm{C}^1([-1,1]) = \operatorname{dom}(A)$ and $Af = f' = g$, so A is closed.

(b) As in (a) the Weierstraß approximation theorem shows that $\mathrm{C}^1([-1,1])$ is dense in $\mathrm{C}([-1,1])$ with respect to $\|\cdot\|_\infty$ and thus, in particular, with respect to $\|\cdot\|_p$. As $\mathrm{C}([-1,1])$ is dense in $L^p(-1,1)$ it follows that the same is true for $\mathrm{C}^1([-1,1]) = \operatorname{dom}(A)$.

To see that A is not closed, let $f \in L^p(-1,1)$ denote the modulus function, i.e. $f(x) = |x|$ for all $x \in [-1,1]$. The sequence (f_k) in $\operatorname{dom}(A)$ given $f_k(x) = \left(x^2 + \frac{1}{k}\right)^{1/2}$ converges uniformly to f, and thus, in particular, with respect to $\|\cdot\|_p$. Moreover,

$$(Af_k)(x) = f_k'(x) = x\left(x^2 + \frac{1}{k}\right)^{-1/2} \quad \text{with} \quad \left|Af_k(x)\right| \le 1$$

for all $x \in [-1,1]$. Since (Af_k) converges pointwise almost everywhere to the signum function, the dominated convergence theorem yields that this convergence also holds in L^p. As $f \notin \operatorname{dom}(A)$, thus A is not closed. □

We wrap up this introductory section with another crucial tool in operator theory.

Definition 3.1.5 (Dual operators)**.** Let X, Y be Banach spaces and let $A\colon \operatorname{dom}(A) \subseteq X \to Y$ be densely defined. The **dual operator** $A'\colon \operatorname{dom}\left(A'\right) \subseteq Y' \to X'$ is defined by

$$\begin{aligned} \operatorname{dom}\left(A'\right) &:= \left\{y' \in Y' \mid \exists\, x' \in X' \colon \left\langle y', Ax\right\rangle = \left\langle x', x\right\rangle \ \forall\, x \in \operatorname{dom}(A)\right\} \\ A'y' &:= x'; \end{aligned}$$

where x' in the second line is the vector that occurs in the definition of $\operatorname{dom}\left(A'\right)$.[1]

We will see more of dual operators – and also their relation to adjoint operators on Hilbert spaces – in Exercise 3.5 and from Chapter 6 onwards.

[1] Note that x' is unique by density of $\operatorname{dom}(A)$ in X.

3.2 Weak derivatives and Sobolev spaces

Example 3.1.4(b) illustrates that differential operators on L^p are not closed when their domain is a space of continuously differentiable functions. The proof showed that, for instance, the modulus function causes such problems: it is not differentiable, but this non-differentiability cannot be "seen" from within an L^p space. To obtain closed differential operators on L^p, one needs a weaker concept of differentiability. This is the topic of the present section. Let us first recall the standard multi-index notation to denote higher order partial derivatives.

Notation 3.2.1. Let $n \in \mathbb{N}$ and let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open. A vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is called a **multi-index**, and its **order** is defined as $|\alpha| := \sum_{j=1}^n \alpha_j$. We write

$$\partial^\alpha f := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$$

for every function $f : \Omega \to \mathbb{R}$ that is continuously differentiable of order $|\alpha|$. With this notation, we have $\partial^{e_j} f = \partial_j f$, where $e_j \in \mathbb{N}_0^n$ denotes the j-th canonical unit vector.

To generalise classical derivatives to a larger class of functions, the following two function spaces are useful.

Definition 3.2.2 (Test functions and local L^p-spaces)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$.

(a) By $\mathrm{C}_\mathrm{c}^\infty(\Omega)$, we denote the space of all **test functions** on Ω, i.e. functions $f : \Omega \to \mathbb{C}$ that are differentiable of every order and vanish outside a compact subset of Ω.

(b) Let $p \in [1, \infty]$. By $L^p_{\mathrm{loc}}(\Omega)$ we denote the space of all Lebesgue measurable $f : \Omega \to \mathbb{C}$ that satisfy $f|_K \in L^p(K)$ for every compact set $\emptyset \neq K \subseteq \Omega$; here we identify functions that are equal almost everywhere on Ω.

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open. Note that for all $p \in [1, \infty]$ one has

$$L^1_{\mathrm{loc}}(\Omega) \supseteq L^p_{\mathrm{loc}}(\Omega) \supseteq L^p(\Omega) + \mathrm{C}(\Omega).$$

In particular, $L^1_{\mathrm{loc}}(\Omega)$ is the largest of all function spaces that we consider on Ω. By integrating – i.e. "testing" – against test functions, one can determine the derivatives of a smooth function.

Proposition 3.2.3. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open.*

(a) *For $f, g \in L^1_{\mathrm{loc}}(\Omega)$ one has $f = g$ if and only if $\int_\Omega f\varphi \,\mathrm{d}x = \int_\Omega g\varphi \,\mathrm{d}x$ for all $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$.*

(b) *Let $\alpha \in \mathbb{N}_0^n$ and $f \in \mathrm{C}^{|\alpha|}(\Omega)$. Then $\partial^\alpha f$ is the unique element of $L^1_{\mathrm{loc}}(\Omega)$ satisfying*

$$\int_\Omega (\partial^\alpha f)\, \varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f\, \partial^\alpha \varphi \,\mathrm{d}x$$

for all $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$.

Proof. (a) This result is sometimes called the fundamental lemma of the calculus of variations. Its proof relies on the fact that there exist sufficiently many test functions on Ω and on techniques from measure theory. For readers interested in the details we provide a proof in supplementary Section 3.A.

(b) Let $f \in C^{|\alpha|}(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$. By extending the function $f\varphi$ by the value 0 outside of Ω, we obtain a function in $C^{|\alpha|}(\mathbb{R}^n)$ that vanishes outside a compact set. For fixed $k \in \{1,\dots,n\}$, (one-dimensional) integration by parts then shows that $\int_\Omega (\partial_k f)\varphi \,\mathrm{d}x = -\int_\Omega f\,\partial_k\varphi \,\mathrm{d}x$. By applying this equality α_1 times for the index $k=1$, then α_2 times for the index $k=2$, and so on, we obtain the required formula. The fact that $\partial^\alpha f$ is the only function in $L^1_{\mathrm{loc}}(\Omega)$ that satisfies this equality follows from (a). □

Proposition 3.2.3(b) characterises the partial derivatives of a function f by integration f against derivatives of test functions. This shows us a path to defining generalised derivatives for a large class of functions. Since properties that rely on testing against functionals are often called **weak properties** in functional analysis, these generalised derivatives are called weak derivatives.

Definition 3.2.4 (Weak derivative)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open, $\alpha \in \mathbb{N}_0^n$ and $f \in L^1_{\mathrm{loc}}(\Omega)$.

(a) We say that f has a **weak αth derivative** if there exists a $g \in L^1_{\mathrm{loc}}(\Omega)$ that satisfies

$$\int_\Omega g\varphi \,\mathrm{d}x = (-1)^\alpha \int_\Omega f\,\partial^\alpha\varphi \,\mathrm{d}x$$

for all $\varphi \in C_c^\infty(\Omega)$. In this case g – which is unique due to Proposition 3.2.3(a)) – is called the **weak αth derivative of** f and is denoted by $g =: \partial^\alpha f$.

As in the classical case, we also use the notation $\partial_j f := \partial^{e_j} f$ for weak derivatives; cf. Notation 3.2.1.

(b) We often just write $\partial^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$ as a shortcut for "f has an αth weak derivative". If $V \subseteq L^1_{\mathrm{loc}}(\Omega)$ is any subset we write $\partial^\alpha f \in V$ as a shortcut for "f has an αth weak derivative and $\partial^\alpha f \in V$."

It follows from Proposition 3.2.3(b) that every function $f \in C^{|\alpha|}(\Omega)$ has a weak αth derivative which coincides with the classical derivative $\partial^\alpha f$. Hence, the notation for weak derivatives is consistent with the notation for classical derivatives. By using weak derivatives we can now fix the issue observed in Example 3.1.4(b) that classical derivatives are not closed operators on L^p-spaces.

Example 3.2.5. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open, $\alpha \in \mathbb{N}_0^n$ and $p \in [1,\infty]$. The differential operator $\partial^\alpha : L^p(\Omega) \supseteq \operatorname{dom}(\partial^\alpha) \to L^p(\Omega)$ is closed when its domain is chosen as large as possible:

$$\begin{aligned}\operatorname{dom}\left(\partial^\alpha\right) &:= \left\{f \in L^p(\Omega) \mid \partial^\alpha f \in L^p(\Omega)\right\} \\ &= \left\{f \in L^p(\Omega) \mid f \text{ has a weak } \alpha\text{th derivative and } \partial^\alpha f \in L^p(\Omega)\right\}\end{aligned}$$

(in the first line of this formula we used the shortcut introduced in Definition 3.2.4(b)). In particular, $\operatorname{dom}(\partial^\alpha)$ is a Banach space when endowed with the graph norm $\|\cdot\|_{\partial^\alpha}$ (or any other graph norm of ∂^α).

Proof. Let (f_k) be a sequence in $\operatorname{dom}(\partial^\alpha)$ that converges in p-norm to a function $f \in L^p(\Omega)$, and assume also that $\partial^\alpha f_k \to g \in L^p(\Omega)$. For every $\varphi \in C_c^\infty(\Omega)$ one has

$$\int_\Omega g\varphi \,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega (\partial^\alpha f_k)\,\varphi \,\mathrm{d}x = \lim_{k\to\infty} (-1)^{|\alpha|} \int_\Omega f_k \partial^\alpha \varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f \partial^\alpha \varphi \,\mathrm{d}x,$$

so f has an αth weak derivative and this derivative is $g \in L^p(\Omega)$. In other words, $f \in \operatorname{dom}(\partial^\alpha)$ and $\partial^\alpha f = g$. The completeness of $\operatorname{dom}(\partial^\alpha)$ with respect to every graph norm of ∂^α is, according to Proposition 3.1.2, a consequence of the closedness of ∂^α. ☐

If one requires $\partial^\alpha f \in L^p(\Omega)$ not only for one multi-index α, but for all α up to a given order, one arrives at the following class of spaces.

Definition 3.2.6 (Sobolev spaces)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open, $p \in [1,\infty]$ and $k \in \mathbb{N}_0$.

(a) The **Sobolev space** of order k with integrability index p is defined as

$$W^{k,p}(\Omega) := \left\{ f \in L^p(\Omega) \mid \partial^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ of order } |\alpha| \le k \right\}$$

and is endowed with the norm

$$\|f\|_{W^{k,p}} := \begin{cases} \left(\sum_{|\alpha|\le k} \|\partial^\alpha f\|_p^p \right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max_{|\alpha|\le k} \|\partial^\alpha f\|_\infty & \text{if } p = \infty. \end{cases}$$

(b) When $p = 2$, we write $H^k(\Omega) := W^{k,2}(\Omega)$ and $\|\cdot\|_{H^k} := \|\cdot\|_{W^{k,2}}$. We endow $H^k(\Omega) = W^{k,2}(\Omega)$ with the inner product[2,3]

$$(f \mid g)_{H^k} = \sum_{|\alpha|\le k} (\partial^\alpha f \mid \partial^\alpha g)_{L^2}.$$

(c) For $p \neq \infty$, we define[4] $W_0^{k,p}(\Omega)$ as the closure of the space $C_c^\infty(\Omega)$ of test functions in $W^{k,p}(\Omega)$. Again, when $p = 2$, we write $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.

Note that one has $W^{0,p}(\Omega) = L^p(\Omega)$. After Example 3.2.5 it should not come as a surprise that the Sobolev spaces are Banach spaces.

Proposition 3.2.7. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open, $p \in [1,\infty]$ and $k \in \mathbb{N}_0$. The Sobolev space $W^{k,p}(\Omega)$ is a Banach space. In particular, $H^k(\Omega)$ is a Hilbert space.*

Proof. For every $\alpha \in \mathbb{N}_0^n$, let $\operatorname{dom}(\partial^\alpha)$ be defined as in Example 3.2.5. Then $W^{k,p}(\Omega) = \bigcap_{|\alpha|\le k} \operatorname{dom}(\partial^\alpha)$ and $\|\cdot\|_{W^{k,p}}$ is equivalent to the norm $\|f\| := \sum_{|\alpha|\le k} \|f\|_{\partial^\alpha}$ on $W^{k,p}(\Omega)$. Since $\operatorname{dom}(\partial^\alpha)$ is complete with respect to the graph norm $\|\cdot\|_{\partial^\alpha}$ (Example 3.2.5), the claim is an immediate consequence of the Lemma 3.2.8 below. ☐

[2] Which is indeed an inner product, as one can easily check, and which induces the norm $\|\cdot\|_{H^k}$.

[3] Throughout the course we will follow the convention from physics that inner products on complex spaces are linear in the second argument (rather than in the first).

[4] We do not need to define the space $W_0^{k,\infty}(\Omega)$ in this course. The curious reader should note that there are (at least) two non-equivalent definitions; see e.g. [Leo09, Remark 11.15].

Lemma 3.2.8. *Let X be a Banach space and let $V_1, \dots, V_n \subseteq X$ be vector subspaces that are Banach spaces with respect to norm $\|\cdot\|_{V_1}, \dots, \|\cdot\|_{V_n}$, respectively. Assume that the inclusion map $(V_k, \|\cdot\|_{V_k}) \to (X, \|\cdot\|_X)$ is continuous for each k. Then $V := V_1 \cap \dots \cap V_n$ is a Banach space with respect to the norm $\|v\|_V := \|v\|_{V_1} + \dots + \|v\|_{V_n}$.*

The proof of the lemma is a small exercise in functional analysis, which we omit. In this section we have seen how to define some closed differential operators on L^p. Closedness of operators is important in spectral theory, as we explain in the next section.

3.3 Spectrum and resolvent

Similarly as for matrices and bounded linear operators on Banach spaces, one can also define spectral values for unbounded operators. However, one must now be careful to always take the domain of the operator into account.

Definition 3.3.1 (Spectrum and resolvent)**.** Let X be a complex Banach space and let $A\colon X \supseteq \operatorname{dom}(A) \to X$ be a closed linear operator.

(a) The **spectrum** of A is the set

$$\sigma(A) := \{\lambda \in \mathbb{C} : \lambda - A\colon \operatorname{dom}(A) \to X \text{ is not bijective}\},$$

where $\lambda - A := \lambda \operatorname{id} - A$ and with $\operatorname{id}\colon \operatorname{dom}(A) \to X$ denoting the inclusion map. The elements of $\sigma(A)$ are called the **spectral values** of A.

(b) The complement $\rho(A) := \mathbb{C} \setminus \sigma(A)$ of the spectrum is called the **resolvent set** of A. For every $\lambda \in \rho(A)$, the **resolvent** of A at λ is the bounded operator $\mathcal{R}(\lambda, A) : X \to \operatorname{dom}(A)$ defined as $\mathcal{R}(\lambda, A) := (\lambda - A)^{-1}$.

Closedness of A was assumed in Definition 3.3.1 for the following reason: Endow $\operatorname{dom}(A)$ with a graph norm; then $\operatorname{dom}(A)$ is a Banach space since A is closed. For $\lambda \in \rho(A)$, the bijection $\lambda - A\colon \operatorname{dom}(A) \to X$ is continuous. By the bounded inverse theorem, the resolvent operator $\mathcal{R}(\lambda, A)$ is continuous from X to $\operatorname{dom}(A)$ and thus, in particular, from X to X since the inclusion map $\operatorname{dom}(A) \to X$ is continuous.

Proposition 3.3.2 (Basic properties of the spectrum and the resolvent)**.** *Let X be a complex Banach space and let $A\colon X \supseteq \operatorname{dom}(A) \to X$ be a closed linear operator.*

(a) *Let $\mu \in \rho(A)$. Each $\lambda \in \mathbb{C}$ with $|\lambda - \mu| < \|\mathcal{R}(\mu, A)\|^{-1}$ satisfies $\lambda \in \rho(A)$ with*

$$\mathcal{R}(\lambda, A) = \sum_{n=0}^{\infty} (\mu - \lambda)^n \mathcal{R}(\mu, A)^{n+1}.$$

In particular, $\rho(A)$ is open in $\mathbb{C}$ and one has $\|\mathcal{R}(\mu, A)\| \geq \frac{1}{\operatorname{dist}(\mu, \sigma(A))}$.

(b) *The resolvent commutes with A, i.e. $A\mathcal{R}(\mu, A)x = \mathcal{R}(\mu, A)Ax$ for each $\mu \in \rho(A)$ and $x \in \operatorname{dom}(A)$.*

(c) *For all $\lambda, \mu \in \rho(A)$, one has the **resolvent identity***

$$\mathcal{R}(\lambda, A) - \mathcal{R}(\mu, A) = (\mu - \lambda)\mathcal{R}(\lambda, A)\mathcal{R}(\mu, A).$$

In particular, the resolvent operators commute.

Proof. (a) Let $\lambda \in \mathbb{C}$ with $|\lambda - \mu| < \|\mathcal{R}(\mu, A)\|^{-1}$. Then $\|(\mu-\lambda)\mathcal{R}(\mu, A)\| < 1$ and so the operator $\mathrm{id} - (\mu - \lambda)\mathcal{R}(\mu, A) \colon X \to X$ is invertible. Thus the identity

$$\lambda - A = \mu - A + \lambda - \mu = \left[\mathrm{id} - (\mu - \lambda)\mathcal{R}(\mu, A)\right](\mu - A)$$

implies that $\lambda \in \rho(A)$ and the claimed series expansion holds. This immediately gives that $\rho(A)$ is open. Moreover, it shows that every $\lambda \in \sigma(A)$ satisfies $|\lambda - \mu| \geq \|\mathcal{R}(\mu, A)\|^{-1}$ and hence, $\mathrm{dist}(\mu, \sigma(A)) \geq \|\mathcal{R}(\mu, A)\|^{-1}$.

(b) Let $\lambda \in \rho(A)$ and $x \in \mathrm{dom}(A)$. Clearly, $(A - \lambda)\mathcal{R}(\lambda, A)x = x = \mathcal{R}(\lambda, A)(A - \lambda)x$ and $\lambda\mathcal{R}(\lambda, A)x = \mathcal{R}(\lambda, A)\lambda x$. Adding those equalities gives the claim.

(c) The resolvent identity can be obtained immediately from the identities

$$\begin{aligned} \mathcal{R}(\lambda, A) &= \mathcal{R}(\lambda, A)\left[\mu\mathcal{R}(\mu, A) - A\mathcal{R}(\mu, A)\right] \\ \text{and} \quad \mathcal{R}(\mu, A) &= \left[\lambda\mathcal{R}(\lambda, A) - A\mathcal{R}(\lambda, A)\right]\mathcal{R}(\mu, A) \end{aligned}$$

which hold for all $\lambda, \mu \in \rho(A)$. ☐

Definition 3.3.3 (Spectral bound)**.** Let X be a complex Banach space and let $A : X \supseteq \mathrm{dom}(A) \to X$ be a closed operator. The **spectral bound** A is defined as

$$\mathrm{s}(A) := \sup\left\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\right\} \in [-\infty, \infty];$$

with the convention $\sup \emptyset = -\infty$.

In general, studying the spectrum of multiplication operators is quite insightful; a simple example is presented in Exercise 3.2. Meanwhile, in the rest of the chapter, we consider the spectrum of a number of concrete differential operators.

Examples 3.3.4 (The spectrum of differential operators on an interval)**.**

(a) Consider the closed operator $A\colon \mathrm{C}([0,1]) \supseteq \mathrm{dom}(A) \to \mathrm{C}([0,1])$, $Af = f'$ that we already studied in Example 3.1.4(a).[5] One has $\sigma(A) = \mathbb{C}$ and thus, $\mathrm{s}(A) = \infty$.

(b) We now consider the operator from (a) on a smaller space: Let $\mathrm{C}_0((0,1])$ denote the space of continuous complex-valued functions on $[0,1]$ that vanish at 0 and let $A\colon \mathrm{C}_0((0,1]) \supseteq \mathrm{dom}(A) \to \mathrm{C}_0((0,1])$, $Af := f'$, where

$$\mathrm{dom}(A) := \left\{f \in \mathrm{C}^1([0,1]) \cap \mathrm{C}_0((0,1]) : f' \in \mathrm{C}_0((0,1])\right\}.$$

Then A is closed, $\sigma(A) = \emptyset$, and thus $\mathrm{s}(A) = -\infty$.

[5] We work on a different interval now, but clearly this does not affect the proof of the closedness of A.

Proof. (a) Let $\lambda \in \mathbb{C}$ and consider the function $f \in C^1([0,1])$, $f(x) = e^{\lambda x}$. Then $(\lambda - A)f = 0$, so $\lambda - A$ is not injective. Hence, $\lambda \in \sigma(A)$.

(b) The closedness of A follows from the closedness of the operator in (a). To show $\sigma(A) = \emptyset$, let $\lambda \in \mathbb{C}$ and $g \in C_0((0,1])$. A function $f : [0,1] \to \mathbb{C}$ is in $\operatorname{dom}(A)$ and solves the equation $(\lambda - A)f = g$ if and only if $f \in C^1([0,1])$ and solves the initial value problem

$$f' = \lambda f - g \qquad \text{and} \qquad f(0) = 0.$$

From the theory of linear ordinary differential equations, $f(x) = -\int_0^x e^{\lambda(x-y)} g(y)\,\mathrm{d}y$ is the unique function with those properties. So $\lambda - A$ is bijective, i.e. $\lambda \in \rho(A)$, and $\mathcal{R}(\lambda, A)g(x) = -\int_0^x e^{\lambda(x-y)} g(y)\,\mathrm{d}y$ for all $g \in C_0((0,1])$. □

As our final example in this section we consider the Laplace operator. For an open set $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ and a function $u \in C^2(\Omega)$, the Laplace operator applied to u is defined as the trace of the Hessian matrix of u:

$$\Delta u := \sum_{j=1}^{n} \partial_j^2 u.$$

We want to define Δ as an operator on $L^p(\Omega)$. To this end, we proceed analogously to Definition 3.2.4 and define Δ in a weak sense.

Definition 3.3.5 (The weak Laplace operator)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ and $u \in L^1_{\mathrm{loc}}(\Omega)$. We say that Δu **exists weakly** or, briefly and by a slight abuse of notation, that $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$, if there exists a (necessarily unique) function $g \in L^1_{\mathrm{loc}}(\Omega)$ that such

$$\int_\Omega g\varphi\,\mathrm{d}x = \int_\Omega u\,\Delta\varphi\,\mathrm{d}x$$

for every test function $\varphi \in C_c^\infty(\Omega)$. In this case we set $\Delta u =: g$.

Similarly as in Example 3.2.5, we could consider the Laplace operator on all functions $u \in L^p(\Omega)$ that satisfy $\Delta u \in L^p(\Omega)$. There are two caveats though. First, the L^p theory turns out to be substantially more involved. We thus stick to the simpler case $p = 2$ for now and return to the general case later. Second, without specifying additional conditions, a similar phenomenon as in Example 3.3.4(a) occurs – the spectrum of the Laplace operator is $\mathbb{C}$ in most cases. A common way to resolve this problem is to impose boundary conditions on the functions in the domain of the operators. Many choices of boundary conditions occur in PDE theory. For now, we focus on one of the simplest cases, which is **Dirichlet boundary conditions**.

Example 3.3.6 (The Dirichlet Laplacian on $L^2(\Omega)$)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open. We define the **maximal Laplace operator** and the **Dirichlet Laplace operator** on $L^2(\Omega)$ by

$$\begin{aligned} \operatorname{dom}(\Delta_{\max}) &:= \{u \in L^2(\Omega) : \Delta u \in L^2(\Omega)\}, & \Delta_{\max} u &:= \Delta u, \\ \operatorname{dom}(\Delta_{\mathrm{Dir}}) &:= \operatorname{dom}(\Delta_{\max}) \cap H_0^1(\Omega), & \Delta_{\mathrm{Dir}} u &:= \Delta u. \end{aligned}$$

They have the following properties:

(a) Both operators $\Delta_{\max}$ and Δ_{Dir} are closed and densely defined.

(b) Let $u, g \in L^2(\Omega)$. Then $u \in \operatorname{dom}(\Delta_{\mathrm{Dir}})$ and $\Delta_{\mathrm{Dir}} u = g$ if and only if $u \in H_0^1(\Omega)$ and[6]

$$\left(v \mid g\right)_{L^2} = -(\nabla v \mid \nabla u)_{L^2} \qquad \text{for all } v \in H_0^1(\Omega).$$

(c) Every $\lambda \in (0, \infty)$ is in the resolvent set $\rho(\Delta_{\mathrm{Dir}})$.

Note that the intersection with $H_0^1(\Omega)$ in the definition of $\operatorname{dom}(\Delta_{\mathrm{Dir}})$ means that we consider only functions that "vanish" on the boundary $\partial\Omega$. A precise formulation of this reasoning requires the theory of boundary traces of Sobolev functions, which we do not discuss in the main text. However, the interested readers can find a brief overview in supplementary Section 3.B, in particular in Theorem 3.B.3.

Proof of Example 3.3.6(a)–(c).

(b) By definition, $u \in \operatorname{dom}(\Delta_{\mathrm{Dir}})$ and $\Delta_{\mathrm{Dir}} u = g$ if and only if $u \in \operatorname{dom}(\Delta_{\max}) \cap H_0^1(\Omega)$ and $\Delta u = g$. The latter is equivalent to $u \in H_0^1(\Omega)$ and $\left(v \mid g\right)_{L^2} = (\Delta v \mid u)_{L^2}$ for all $v \in C_c^\infty(\Omega)$, since a function $v \in C_c^\infty(\Omega)$ if and only if its complex conjugate $\overline{v} \in C_c^\infty(\Omega)$.

Note that for each $u \in H_0^1(\Omega)$, we have $(\Delta v \mid u)_{L^2} = -(\nabla v \mid \nabla u)_{L^2}$ for all $v \in C_c^\infty(\Omega)$ by Definition 3.2.4(a). Whence $u \in \operatorname{dom}(\Delta_{\mathrm{Dir}})$ and $\Delta_{\mathrm{Dir}} u = g$ if and only if $u \in H_0^1(\Omega)$ and

$$\left(v \mid g\right)_{L^2} = -(\nabla v \mid \nabla u)_{L^2} \qquad \text{for all } v \in C_c^\infty(\Omega).$$

However, the validity of the above for all $v \in C_c^\infty(\Omega)$ is equivalent to its validity for all $v \in H_0^1(\Omega)$, since both sides are continuous in v with respect to the H^1-norm and since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ with respect to this norm.

(a) Both operators $\Delta_{\max}$ and Δ_{Dir} are densely defined since their domains contain the dense subspace $C_c^\infty(\Omega)$. The fact that $\Delta_{\max}$ is closed follows by precisely the same argument as the closedness of the differential operator ∂^α in Example 3.2.5 (but using Definition 3.3.5 in place of Definition 3.2.4).

Substituting $v := u$ in the equality in (b), we observe that[7,8]

$$\|\nabla u\|_{L^2}^2 \le \|u\|_{L^2} \, \|\Delta u\|_{L^2} \tag{3.3.1}$$

from the Cauchy–Schwarz inequality in $L^2(\Omega)$.

Since $\Delta_{\max}$ is closed, $\operatorname{dom}(\Delta_{\max})$ is complete with respect to the graph norm $\|u\|_{\Delta_{\max}} = \|\Delta u\|_2 + \|u\|_2$. As the Sobolev space $H_0^1(\Omega)$ is also complete (by Proposition 3.2.7), it follows from Lemma 3.2.8 that the norm $u \mapsto \|\Delta u\|_2 + \|\nabla u\|_2 + \|u\|_2$ is complete on $\operatorname{dom}(\Delta_{\mathrm{Dir}})$. Moreover, this norm is equivalent to every graph norm of Δ_{Dir} since the inequality (3.3.1) implies that $\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2 \le (\|\Delta u\|_2 + \|u\|_2)^2$ for each $u \in \operatorname{dom}(\Delta_{\mathrm{Dir}})$. Hence, Δ_{Dir} is also closed.

[6] For $v \in H^1(\Omega)$ we use the same notation $\nabla v := (\partial_1 v, \ldots, \partial_n v)^{\mathrm{T}} \in L^2(\Omega; \mathbb{C}^n)$ as for classical derivatives.
[7] Here $\|\nabla u\|_{L^2}^2 := \int_\Omega \|\nabla v(x)\|_2^2 \, dx$, where $\|\nabla v(x)\|_2$ is the Euclidean norm of the vector $\nabla v(x) \in \mathbb{C}^n$.
[8] This is a very simple example of an **interpolation inequality**.

(c) Let $\lambda \in (0,\infty)$ and let $g \in L^2(\Omega)$. For every $u \in L^2(\Omega)$ the conditions $u \in \operatorname{dom}(\Delta_{\mathrm{Dir}})$ and $(\lambda - \Delta_{\mathrm{Dir}})u = g$ are – according to (b) – equivalent to $u \in H^1_0(\Omega)$ and $\big(v \mid \lambda u - g\big)_{L^2} = -(\nabla v \mid \nabla u)_{L^2}$ for all $v \in H^1_0(\Omega)$, which is in turn equivalent to $u \in H^1_0(\Omega)$ and

$$\lambda\,(v \mid u)_{L^2} + (\nabla v \mid \nabla u)_{L^2} = \big(v \mid g\big)_{L^2} \qquad \text{for all } v \in H^1_0(\Omega). \tag{3.3.2}$$

Observe that for every $u \in H^1_0(\Omega)$ one has

$$\min\{1,\lambda\}\big(\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}\big) \le \lambda\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2} \le \max\{1,\lambda\}\big(\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}\big).$$

Hence the norm induced by the inner product

$$\mathfrak{a}(v,u) := \lambda\,(v \mid u)_{L^2} + (\nabla v \mid \nabla u)_{L^2}$$

is equivalent to the standard H^1-norm on $H^1_0(\Omega)$. As $\big(\cdot \mid g\big)_{L^2}$ is a continuous antilinear functional[9] on $H^1_0(\Omega)$, the Riesz representation theorem on Hilbert spaces gives a unique $u \in H^1_0(\Omega)$ satisfying (3.3.2). Consequently, $\lambda \in \rho(\Delta_{\mathrm{Dir}})$. □

The spectral information given in Example 3.3.6(c) is far from optimal. We shall see more about the spectrum of Δ_{Dir} as we proceed.

[9] Recall that a map $\varphi\colon X \to \mathbb{C}$ from a complex Banach space X to $\mathbb{C}$ is called **antilinear** if $\varphi(x+\alpha y) = \varphi(x) + \overline{\alpha}\varphi(y)$ for all $x, y \in X$ and all $\alpha \in \mathbb{C}$.

Exercises for Chapter 3

Exercise 3.1 (The derivative at a point is not closed)**.** Define the operator

$$A\colon \, \mathrm{C}([0,1]) \supseteq \mathrm{dom}\,(A) := \mathrm{C}^1([0,1]) \to \mathbb{C}, \qquad Af := f'(1).$$

Prove that A is densely defined but not closed.

Exercise 3.2 (Multiplication operators)**.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open and let $\mathrm{C}_0(\Omega)$ denote the space of continuous functions $f : \Omega \to \mathbb{C}$ with the following property: for every $\varepsilon > 0$ there exists a compact set $K \subseteq \Omega$ such that $\left|f(x)\right| \leq \varepsilon$ for all $x \in \Omega \setminus K$. This is a Banach space with respect to the sup norm $\|\cdot\|_\infty$.

Let $h : \Omega \to \mathbb{C}$ be continuous and define the operator M_h on $\mathrm{C}_0(\Omega)$ by

$$\mathrm{dom}\,(M_h) := \left\{f \in \mathrm{C}_0(\Omega) : hf \in \mathrm{C}_0(\Omega)\right\}, \qquad M_h f := hf.$$

(a) Show that M_h is closed and densely defined.

(b) Prove that M_h is everywhere defined if and only if h is bounded.

(c) Show that $\sigma(M_h) = \overline{h(\Omega)}$.

Exercise 3.3. This exercise applies the closed graph theorem (Theorem 3.1.3).

(a) Let X, Y, Z be Banach spaces and consider linear operators $X \xrightarrow{T} Y \xrightarrow{J} Z$, where J is injective. Show that if J and JT are continuous, then so is T.

(b) Let (Ω, μ) be a finite measure space and $1 \leq p \leq q \leq \infty$. Let $T\colon L^p(\Omega,\mu) \to L^p(\Omega,\mu)$ be a bounded linear operator whose range is contained in $L^q(\Omega,\mu)$. Show that T is continuous from $L^p(\Omega,\mu)$ to $L^q(\Omega,\mu)$.

(c) Let X be a Banach space, let $A\colon X \supseteq \mathrm{dom}\,(A) \to X$ be a closed linear operator and $T\colon X \to X$ a bounded linear operator such that $T(X) \subseteq \mathrm{dom}\,(A)$. Show that T is continuous from X to $\mathrm{dom}\,(A)$ if $\mathrm{dom}\,(A)$ is endowed with a graph norm.

Exercise 3.4.

(a) Consider the function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ given by $f(x) = |x|$. Show that f is weakly differentiable and compute its weak derivative.

(b) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathrm{C}^1(\mathbb{R}^n \setminus \{0\})$[10] and assume that there exist $g_1, \dots g_n \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ coinciding respectively with the classical partial derivatives $\partial_1 f, \dots, \partial_n f$ on $\mathbb{R}^n \setminus \{0\}$. Assume in addition that $\int_{\|x\|_2 = r} f(x) \, \mathrm{d}\sigma(x) \to 0$ as $r \downarrow 0$, where σ denote the surface measure of the ball with radius r in $\mathbb{R}^n$.

Show that f is weakly differentiable with weak derivatives $g_1, \dots, g_n$.

Hint: Given a test function $\varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$, let $R > 0$ be such that $\operatorname{supp}\varphi \subseteq B_{<R}(0)$. Apply the divergence theorem on a 'shell' $\{x \in \mathbb{R}^n : r \le \|x\|_2 \le R\}$, and let $r \downarrow 0$.

(c) Let $f \colon \mathbb{R}^n \to \mathbb{R}$, $f(x) = \|x\|_2$. Show that f is weakly differentiable and compute its weak derivatives $\partial_1 f, \dots, \partial_n f$.

Exercise 3.5. Let X, Y be Banach spaces over the same field and let $A \colon X \supseteq \operatorname{dom}(A) \to Y$ be a densely defined linear operator.

(a) Prove that the dual operator $A' \colon Y' \supseteq \operatorname{dom}(A') \to X'$ (Definition 3.1.5) is closed.

(b) Give an example where A' is not densely defined.

Suggestion: Take $X = Y = \ell^1$ and take A to be a multiplication operator, analogous to Exercise 3.2.

(c) If $X = Y$, show that $\operatorname{dom}((\lambda + A)') = \operatorname{dom}(A')$ and $\operatorname{dom}((\alpha A)') = \operatorname{dom}(A')$ and that

$$(\lambda + A)' = \lambda + A' \qquad \text{and} \qquad (\alpha A)' = \alpha A'$$

for all scalars λ, α with $\alpha \neq 0$. What goes wrong if $\alpha = 0$?

Assume now that $X = Y$, that the scalar field is $\mathbb{C}$, and that A is closed.

(d) Show that if $\lambda \in \rho(A)$, then also $\lambda \in \rho(A')$ and $\mathcal{R}(\lambda, A') = \mathcal{R}(\lambda, A)'$, where the latter operator denotes the dual of $\mathcal{R}(\lambda, A) \in \mathcal{L}(X)$.

(e) Conversely, show that if $\lambda \in \rho(A')$, then also $\lambda \in \rho(A)$.

Hints: First show that $\|x\|_X \le \|(\lambda - A)x\|_X \left\|\mathcal{R}(\lambda, A')\right\|$ for all $x \in \operatorname{dom}(A)$. Then derive that there exists a (λ-dependent) number $c > 0$ such that $\|x\|_A \le c\,\|(\lambda - A)x\|_X$ for all $x \in \operatorname{dom}(A)$. Hence deduce that $\lambda - A$ is injective and has closed range.

[10] Strictly speaking, this means that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and f has a representative whose restriction to the set $\mathbb{R}^n \setminus \{0\}$ is C^1.

Notes for Chapter 3

Unbounded operators and their spectra

> It is a fact of life that many of the most important operators which occur in mathematical physics are not bounded. — [RS80, p. 249]

As the above quote from the classic text of Reed and Simon suggests, unbounded operators are among the most fundamental objects in functional analysis. Their importance was already recognised in the early days of quantum mechanics: the so-called **position operator** that describes – no terminological surprise here – the position of a particle, acts as a multiplication with an unbounded function on the space $L^2(\mathbb{R})$. And the so-called **momentum operator** acts as a first order differential operator on the same space; see [Hal13, Chapter 3] for an accessible introduction to the relevant physical concepts, tailored to mathematicians. Similarly, motivated by the study of differential operators arising in (classical and quantum) physics, the spectral theory of unbounded operators has long been a fruitful subject. For applications in quantum physics, we again refer the interested reader to [Hal13] and [RS80, Chapter VIII].

While we are in the realm of physics, we point out that the expression $(\nabla v \mid \nabla u)_{L^2}$ plays an important role in the study of the Laplacian in the weak formulation, as shown in Example 3.3.6(b). The map $(v,u) \mapsto (\nabla v \mid \nabla u)_{L^2}$ is a **sesquilinear form**, and the associated **quadratic form** $u \mapsto \|\nabla u\|_{L^2}^2$ often has the physical interpretation of an 'energy'. Sesquilinear form methods provide another approach to the study of differential operators, and unsurprisingly are widely used in mathematical physics and the calculus of variations. We will discuss some basic aspects of these methods in Chapter 5.

A word on the closed graph theorem

The usual proofs of the closed graph theorem for Banach spaces (Theorem 3.1.3), found in many standard texts on functional analysis, rely on the **Baire category theorem**. It turns out that it is not necessary to use this theorem, as shown, for instance, in [Kes21]; cf. [Kes17]. In addition, the latter reference illustrates the equivalence between closed graph theorem, open mapping theorem, bounded inverse theorem, and the uniform boundedness principle in Banach spaces. This shows that completeness is the underlying principle in these foundational results.

Weak derivatives and Sobolev spaces

An alternative but quite natural way to define a space of weakly differentiable functions is via limits of classically differentiable functions. To be precise, we define the norm $\|\cdot\|_{W^{k,p}}$ exactly as in Definition 3.2.6, and let $H^{k,p}(\Omega)$ denote the completion of the space

$$\left\{u \in C^{\infty}(\Omega) : \|u\|_{W^{k,p}} < \infty\right\}$$

with respect to the $W^{k,p}$ norm. It is straightforward to check that $H^{k,p}(\Omega) \subseteq W^{k,p}(\Omega)$, but this leaves the obvious question of whether the converse inclusion holds. It took some time before this problem was finally resolved in the affirmative in 1964 by Norman Meyers and James Serrin in their iconic paper [MS64]. We say a few more words about this in the Supplement 3.A.

The weak derivative in Definition 3.2.4 is often also called the **distributional derivative**. This is a fundamental notion in the theory of distributions, also known as generalised functions. The idea of extending differentiation beyond the classical setting originates well before the 20th century; a detailed historical account of this topic can be found, for example, in [Lue82, Chapter 2]. From the perspective of the general theory of distributions, elements of a Sobolev space are simply 'well-behaved' distributions, where the element itself and all its distributional derivatives up to a given order can be represented by L^p functions.

One can also define Sobolev spaces via the Fourier transform, as is typically done in harmonic analysis. For readers interested in this approach, there is a wide selection of good literature, including [Str03, Chapter 8], [Gru09, Part II], and [Gra14, Chapter 1].

Encore: If you want to know more...

3.A Regularisation of functions

In this supplementary section, we use the following notation: given a measurable function $h : \mathbb{R}^n \to \mathbb{R}$, for each $t > 0$ we set

$$h_t(x) := \frac{1}{t^n} h\left(\frac{x}{t}\right). \tag{3.A.1}$$

Definition 3.A.1 (Mollifiers)**.** A **mollifier** is a family $\{\rho^t : t > 0\}$ of functions $\rho^t : \mathbb{R}^n \to \mathbb{R}$ satisfying the following properties for each $t > 0$:

(1) $\rho^t \in C_c^\infty(\mathbb{R}^n)$ and ρ^t is supported in $B_{\le t}(0)$; and

(2) $\rho^t \ge 0$ on $\mathbb{R}^n$ and $\int_{\mathbb{R}^n} \rho^t(x)\,dx = 1$.

We define a special test function $\theta \in C_c^\infty(\mathbb{R}^n)$ by

$$\theta(x) := \begin{cases} c\exp\left(-\frac{1}{1-|x|^2}\right) & |x| < 1 \\ 0 & |x| \ge 1, \end{cases} \tag{3.A.2}$$

where the constant $c > 0$ is chosen so that $\int_{\mathbb{R}^n} \theta(x)\,dx = 1$. Then the family $\{\theta_t : t > 0\}$ (using the notation (3.A.1)) is called the **standard mollifier**.

The key features of mollifiers are that they consist of very smooth functions, and most crucially, by taking $t \downarrow 0$, the support of ρ^t can be made as small as desired. This allows us to **regularise** non-smooth functions via convolutions while maintaining precise control of the support. We summarise some standard facts about convolutions and regularisations below, and refer to the literature (e.g. [Bre11, Section 4.4]) for the proofs.

Proposition 3.A.2 (An analysis toolkit)**.**

(a) (Young's inequality) *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$. Then $f * g \in L^p(\mathbb{R}^n)$ with* $\operatorname{supp}(f * g) \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g}$, *and*

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

(b) (Regularisation) *For all $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$, one has $\varphi * g \in C^\infty(\mathbb{R}^n)$ with*

$$\partial^\alpha(\varphi * g) = (\partial^\alpha \varphi) * g \quad \textit{for all } \alpha \in \mathbb{N}_0^n.$$

(c) (Approximate identity) *Let* $\{\rho^t : t > 0\}$ *be a mollifier. Then* $\lim_{t\to 0} \|\rho^t * f - f\|_p = 0$ *for all* $f \in L^p(\mathbb{R}^n)$ *with* $1 \le p < \infty$.

(d) (Density of test functions) *For every open set* $\emptyset \ne \Omega \subseteq \mathbb{R}^n$, *the space of test functions* $C_c^\infty(\Omega)$ *is dense in* $L^p(\Omega)$ *for* $1 \le p < \infty$.

We can now complete the proof of Proposition 3.2.3 from the main text.

Theorem 3.A.3 (Fundamental lemma of the calculus of variations)**.** *Let* $\emptyset \ne \Omega \subseteq \mathbb{R}^n$ *be open. Suppose* $f \in L^1_{\mathrm{loc}}(\Omega)$ *satisfies*

$$\int_\Omega f\varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{3.A.3}$$

Then $f = 0$ *in* Ω.

Proof. Choose increasing compact subsets Ω_k so that $\Omega = \bigcup_{k\in\mathbb{N}} \Omega_k$; for instance, take

$$\Omega_k := \left\{ x \in \Omega : |x| \le k \text{ and } \mathrm{dist}(x, \partial\Omega) \ge \frac{1}{k} \right\}.$$

It suffices to prove that $f(x) = 0$ for almost every $x \in \Omega_k$ for each $k \in \mathbb{N}$. We do this via an approximation argument.

Since $f \in L^1_{\mathrm{loc}}(\Omega)$, we have $f_k := f\,\mathbb{1}_{\Omega_k} \in L^1(\Omega)$ for each $k \in \mathbb{N}$. Denote by $\tilde{f}_k$ the extension of f_k by 0 outside Ω_k, and note that $\tilde{f}_k \in L^1(\mathbb{R}^n)$. Since Ω_k is contained strictly in Ω, for all sufficiently small $t > 0$ (depending on k) the support of $\theta_t(x - \cdot)$ lies in Ω for all $x \in \Omega_k$, and therefore $\theta_t(x - \cdot) \in C_c^\infty(\Omega)$ for all $x \in \Omega_k$. Consequently

$$(\theta_t * \tilde{f}_k)(x) = \int_{\mathbb{R}^n} \tilde{f}_k(y)\theta_t(x-y)\,\mathrm{d}y = \int_\Omega f_k(y)\theta_t(x-y)\,\mathrm{d}y = 0$$

for all sufficiently small $t > 0$ and all $x \in \Omega_k$, where we have used assumption (3.A.3) in the last equality. By Proposition 3.A.2(c), we conclude

$$0 = \lim_{t\downarrow 0}(\theta_t * \tilde{f}_k) = \tilde{f}_k$$

in $L^1(\mathbb{R}^n)$, which implies that $f_k = 0$ as desired. □

The following theorem relies on a clever use of partition of unity and regularisation.

Theorem 3.A.4 (Meyers, Serrin)**.** *Let* $\emptyset \ne \Omega \subseteq \mathbb{R}^n$ *be open, and let* $k \in \mathbb{N}$ *and* $p \in [1, \infty)$. *Then* $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ *is dense in* $W^{k,p}(\Omega)$.

The original 1964 paper of Meyers and Serrin [MS64] arguably has one of the most iconic titles in the field of analysis: $H = W$. It is merely one-and-a-half pages long with an extremely brief proof. The reader interested in a detailed proof is thus advised to consult more recent literature, e.g. [GT01, Theorem 7.9] or [Leo09, Theorem 11.24]. Having said that, the following simple yet important corollary is worth presenting in detail.

Corollary 3.A.5. *Let $k \in \mathbb{N}$ and $p \in [1,\infty)$. Then*

$$W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n).$$

Proof. The non-trivial inclusion to prove is $W^{k,p}(\mathbb{R}^n) \subseteq W_0^{k,p}(\mathbb{R}^n)$, i.e. to show that every $u \in W^{k,p}(\mathbb{R}^n)$ can be approximated in the $W^{k,p}$ norm by a sequence $(u_m) \subset \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)$.

By the Meyers-Serrin Theorem (Theorem 3.A.4), it suffices to prove the claim for $u \in \mathrm{C}^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$. For this purpose, fix a function $\zeta \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)$ such that $0 \le \zeta \le 1$ on $\mathbb{R}^n$ and $\zeta \equiv 1$ for $|x| \le 1$, and set

$$u_m(x) := \zeta\left(\frac{x}{m}\right) u(x), \qquad x \in \mathbb{R}^n$$

for $m \in \mathbb{N}$. Clearly $u_m \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)$ for each $m \in \mathbb{N}$, and the dominated convergence theorem yields $u_m \to u$ in $L^p(\mathbb{R}^n)$. The generalised Leibniz rule yields

$$\partial^\alpha u_m = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta\big(\zeta(\cdot/m)\big)\partial^{\alpha-\beta} u = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \frac{1}{m^{|\beta|}} (\partial^\beta \zeta)(\cdot/m)\partial^{\alpha-\beta} u$$

for every multi-index $|\alpha| \le k$. If $\beta = (0,\dots,0)$, then again by dominated convergence we obtain $\zeta(\frac{\cdot}{m})\partial^\alpha u \to \partial^\alpha u$ in $L^p(\mathbb{R}^n)$. For $\beta \neq 0$, observe that

$$\int_{\mathbb{R}^n} \left|\partial^\beta\big(\zeta(x/m)\big)\partial^{\alpha-\beta} u(x)\right|^p \,\mathrm{d}x \le \frac{C^p}{m^{|\beta| p}} \int_{\mathbb{R}^n} \left|\partial^{\alpha-\beta} u(x)\right|^p \,\mathrm{d}x \to 0$$

as $m \to \infty$, where $C := \max_{|\beta| \le k} \|\partial^\beta \zeta\|_{L^\infty(\mathbb{R}^n)} < \infty$. Altogether, we have shown that $\partial^\alpha u_m \to \partial^\alpha u$ in L^p for every multi-index $|\alpha| \le k$, and thus $u_m \to u$ in $W^{k,p}(\mathbb{R}^n)$. □

3.B Traces of $W^{1,p}$ functions

In Example 3.3.6, we introduced the Dirichlet Laplace operator Δ_{Dir} with domain contained in $H_0^1(\Omega)$, a space which intuitively encodes the boundary condition '$u = 0$ on $\partial\Omega$'. In Theorem 3.B.3 below, we make precise the sense in which a function in the Sobolev space $W_0^{1,p}(\Omega)$ 'vanishes' on the boundary $\partial\Omega$. This is achieved via the theory of **traces**. Before we proceed, we need to understand 'regularity' of the boundary of subsets of $\mathbb{R}^n$.

A **rigid motion** of $\mathbb{R}^n$ is a map $T : \mathbb{R}^n \to \mathbb{R}^n$ of the form $T(x) = Rx + c$, for a rotation R and fixed $c \in \mathbb{R}^n$. Using this notion, we can say what it means for the boundary of an open set $\Omega \subseteq \mathbb{R}^n$ to be Lipschitz continuous. Intuitively, this means that the boundary looks locally like the graph of a scalar-valued Lipschitz function in $n-1$ variables.

Definition 3.B.1. Let $n \ge 2$, and let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be open. We say that the boundary $\partial\Omega$ is **Lipschitz continuous** (or simply **Lipschitz**) if for every $\xi_0 \in \partial\Omega$, there exists a rigid motion T with $T(\xi_0) = 0$, a Lipschitz continuous function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, and $r > 0$ such that

$$T(\Omega \cap B(\xi_0, r)) = \left\{x \in B(0,r) \,:\, x_n > f(x_1,\dots,x_{n-1})\right\}$$

where the **local coordinates** are given by $x := T(\xi)$ for all $\xi \in \Omega \cap B(\xi_0, r)$.

Likwise, given $k \in \mathbb{N}_0 \cup \{\infty\}$, we say that $\partial\Omega$ is **of class** C^k if the functions f occurring above belong to $\mathrm{C}^k(\mathbb{R}^{n-1};\mathbb{R})$.

An example of a region in $\mathbb{R}^2$ with Lipschitz boundary is illustrated in Figure 3.B.1. In practice, note that the open balls $B(\xi_0, r)$ may be replaced by other kinds of open sets, e.g. open cubes, according to what is most convenient.

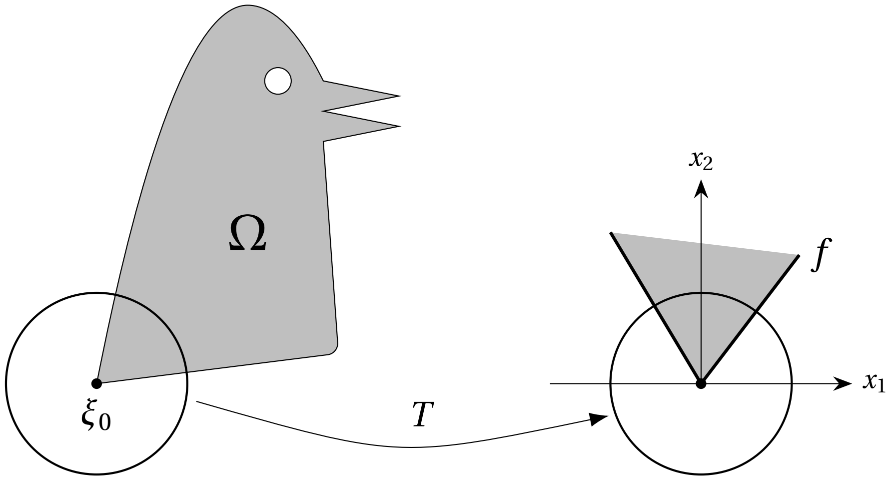

Figure 3.B.1: A region with Lipschitz boundary and local coordinates around ξ_0.

In the theorem below, the boundary $\partial\Omega$ is equipped with the $(d-1)$-dimensional Hausdorff measure.

Theorem 3.B.2 (Trace operator)**.** *Let $n \geq 2$, $1 \leq p < \infty$, and let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be an open set with bounded and Lipschitz continuous boundary $\partial\Omega$. There exists a unique linear operator* $\mathrm{Tr}\colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$, *called the **trace operator**, with the following properties:*

(a) $\mathrm{Tr}\, u = u|_{\partial\Omega}$ *for all* $u \in W^{1,p}(\Omega) \cap \mathrm{C}(\overline{\Omega})$.

(b) (Trace inequality) *There exists $C > 0$ (depending only on Ω) such that*

$$\|\mathrm{Tr}\, u\|_{L^p(\partial\Omega)} \leq C\, \|u\|_{W^{1,p}(\Omega)}$$

for all $u \in W^{1,p}(\Omega)$.

For a proof, we refer to [Leo09, Theorem 18.1]. Note that in this reference, the result is stated more generally for open sets with unbounded boundary, but it is easy to see that it reduces to our statement when $\partial\Omega$ is bounded.

The space $W_0^{1,p}(\Omega)$ can now be characterised as the kernel of the trace operator.

Theorem 3.B.3 (Characterisation of $W_0^{1,p}(\Omega)$)**.** *Let $n \geq 2$, $1 \leq p < \infty$, and let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be an open set with bounded and Lipschitz continuous boundary $\partial\Omega$. For all $u \in W^{1,p}(\Omega)$, the following assertions are equivalent:*

(i) $\mathrm{Tr}\, u = 0$.

(ii) $u \in W_0^{1,p}(\Omega)$.

We refer to [Leo09, Theorem 18.7] for the proof.

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