Chapter 2

Eventual positivity in finite dimensions

In this chapter, we encounter our main protagonist, eventual positivity (Section 2.2). We shall see that many spectral properties of positive matrix semigroups remain true for eventually positive ones, but to prove this, we first need more advanced tools from spectral theory (Section 2.1). Remarkably though, already in the finite dimensional setting, there are significant differences between the positive and the eventually positive case. The characterisation of eventual positivity of a matrix semigroup $(e^{tA})_{t\geq 0}$ in terms of A (Section 2.3) has a different flavour than for positive semigroups, and the differences become even clearer when it comes to perturbation theory (Section 2.4).

2.1 Prelude: Spectral decomposition of matrices

For the analysis of matrix powers and exponentials in Exercise 1.4, you have already encountered a very useful tool: the Jordan normal form. In this section, the same tool is used to derive a variety of spectral properties, so let us fix the notation for it.

Let $A \in \mathbb{C}^{n\times n}$. By a coordinate transform, one can achieve that A is in **Jordan normal form**. This means that A can be written in block diagonal form as

$$A = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix} \tag{2.1.1}$$

for matrices $J_k \in \mathbb{C}^{n_k\times n_k}$ that are called **Jordan blocks**, i.e. each of them has the form

$$J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix} \tag{2.1.2}$$

for a number $\lambda_k \in \mathbb{C}$. The numbers $\lambda_1, \dots, \lambda_m$ are the eigenvalues of A, where the number of occurrences of each eigenvalue in this list coincides with its geometric multiplicity. In other words, the geometric multiplicity of λ_k is the number of Jordan blocks associated to λ_k. On the other hand, the algebraic multiplicity of λ_k is the sum of the sizes of all Jordan blocks corresponding to λ_k. We set

$$N_k := J_k - \lambda_k = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix},$$

which is nilpotent of index n_k.

For an open set $\emptyset \neq \Omega \subseteq \mathbb{C}$, a point $\lambda \in \Omega$, and an analytic function $f\colon \Omega \setminus \{\lambda\} \to \mathbb{C}^{m \times n}$, the point λ is called a **pole** of f if for some $p \in \mathbb{N}$, the $\lim_{\mu \to \lambda} (\mu - \lambda)^p f(\mu)$ exists and is non-zero. In this case, p is unique and is called the **pole order** of λ. By considering the entries of f, one can see that a similar Laurent series expansion as in the scalar-valued case continues to hold.

Proposition 2.1.1 (Eigenvalues are poles of the resolvent)**.** *Let $\lambda \in \mathbb{C}$ be an eigenvalue of a matrix $A \in \mathbb{C}^{n \times n}$. If the largest Jordan block of A corresponding to λ has size $p \geq 1$, then*

(a) *λ is a pole of the resolvent $\mathcal{R}(\cdot, A)$ with pole order p, i.e. there exist matrices $Q_j \in \mathbb{C}^{n \times n}$ for $j \geq -p+1$ such that $Q_{-p+1} \neq 0$ and*

$$(\mu - A)^{-1} = \sum_{j=-p}^{\infty} Q_{j+1} (\mu - \lambda)^j$$

for all $\mu \neq \lambda$ that are sufficiently close to λ.

(b) *The range of Q_{-p+1} is contained in the eigenspace* $\ker(\lambda - A)$.

Proof. (a) We may assume that A is in Jordan normal form (2.1.1). Thus, in order to compute the inverse $(\mu - A)^{-1}$, we only need to compute the inverses $(\mu - J_k)^{-1}$ of the Jordan blocks $J_k \in \mathbb{C}^{n_k \times n_k}$ of A. Due to the Neumann series representation of the resolvent from Proposition 1.2.7(b), we obtain for all $\mu \in \rho(A)$ and all indices k,

$$(\mu - J_k)^{-1} = (\mu - \lambda_k)^{-1} \left(\mathrm{id} - (\mu - \lambda_k)^{-1} N_k\right)^{-1} = \sum_{j=0}^{n_k - 1} \frac{N_k^j}{(\mu - \lambda_k)^{j+1}}. \tag{2.1.3}$$

For $\lambda \neq \lambda_k$, the series in (2.1.3) is holomorphic near λ, so one gets a Taylor expansion of $(\mu - J_k)^{-1}$ about the point λ. In particular, only blocks with $\lambda = \lambda_k$ can produce poles at λ the formula for $(\mu - J_k)^{-1}$ immediately gives the coefficients of the Laurent series expansion of $(\mu - J_k)^{-1}$ about the point λ and contributes a pole of order n_k. Thus, λ is a pole of order $\max_k n_k = p$.

We infer from (2.1.3) that the coefficient of $(\mu - \lambda)^{-p}$ in $(\mu - J_k)^{-1}$ is $N_k^{p-1} \neq 0$ whenever $n_k = p$ and 0 if $n_k < p$ or $\lambda \neq \lambda_k$. Hence, the blocks in Q_{-p+1} corresponding to $n_k = p$ and $\lambda_k = \lambda$ are non-zero. In particular, $Q_{-p+1} \neq 0$.

(b) For each index k, $\operatorname{rg}(N_k^{p-1}) \subseteq \ker(\lambda_k - J_k)$ because

$$(\lambda_k - J_k)N_k^{p-1} = -N_k^p = 0.$$

From the proof of (a), we know that the non-zero blocks in Q_{-p+1} are exactly those for which $\lambda_k = \lambda$ and $n_k = p$. Consequently, $\operatorname{rg} Q_{-p+1}$ is contained in $\ker(\lambda - A)$. ☐

The formula (2.1.3) has a number of useful consequences that we discuss now. We need the following crucial observation about contour integration in complex analysis.

Proposition 2.1.2. *Let $\lambda \in \mathbb{C}$ and let γ be a closed C^1-path in $\mathbb{C} \setminus \{\lambda\}$.*

(a) *For each integer $j \neq 1$ one has $\oint_\gamma \frac{1}{(\mu-\lambda)^j} \,\mathrm{d}\mu = 0$.*

(b) *If γ encircles λ precisely once, then $\frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{1}{\mu-\lambda} \,\mathrm{d}\mu = 1$.*

(c) *If γ does not encircle λ, then $\oint_\gamma \frac{1}{\mu-\lambda} \,\mathrm{d}\mu = 0$.*

Proposition 2.1.2 can be applied to every entry of the $(\mu - J_k)^{-1}$ in formula (2.1.3). Doing this for all the Jordan blocks of A, one immediately obtains the following.

Corollary 2.1.3. *Let $A \in \mathbb{C}^{n\times n}$ and let γ be a closed C^1-path in $\mathbb{C} \setminus \sigma(A)$.*

(a) *If γ encircles each eigenvalue of A precisely once, then $\frac{1}{2\pi \mathrm{i}} \oint_\gamma (\mu - A)^{-1} \,\mathrm{d}\mu = \mathrm{id}$.*

(b) *If γ does not encircle any eigenvalue of A, then $\oint_\gamma (\mu - A)^{-1} \,\mathrm{d}\mu = 0$.*

Corollary 2.1.3 now comes in handy to prove the representation formula, and hence the uniqueness, in the following proposition.

Proposition 2.1.4 (Spectral decomposition)**.** *Let $A \in \mathbb{C}^{n\times n}$ and let $\sigma_0 \subseteq \sigma(A)$. There exists precisely one projection $P \in \mathbb{C}^{n\times n}$ that has the following properties:*

(a) *P commutes with A.*

(b) *The restrictions of A to the range and the kernel of P have the spectra*

$$\sigma\left(A|_{\operatorname{rg} P}\right) = \sigma_0 \qquad \text{and} \qquad \sigma\left(A|_{\ker P}\right) = \sigma(A) \setminus \sigma_0.$$

Moreover, P is given by the formula

$$P = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (\mu - A)^{-1} \,\mathrm{d}\mu \tag{2.1.4}$$

for any closed C^1-path γ in $\mathbb{C}$ that encircles each element of σ_0 precisely once, but no element of $\sigma(A) \setminus \sigma_0$.

Proof. Existence: After a coordinate transform, we may assume that A is in Jordan normal form, i.e. that it is given by the formula (2.1.1); we use the notation specified next to this formula. After ordering the eigenvalues $\lambda_1, \dots, \lambda_m$ appropriately, we can find an $\ell \in \{0, 1, \dots, m\}$ such that $\sigma_0 = \{\lambda_k : 1 \le k \le \ell\}$ and $\sigma(A) \setminus \sigma_0 = \{\lambda_k : \ell < k \le m\}$; we allow $\ell = 0$ to account for the case $\sigma_0 = \emptyset$. Now, let $P \in \mathbb{C}^{n \times n}$ be the projection onto the first $n_1 + \dots + n_\ell$ components of $\mathbb{C}^n$. Then P has the required properties.

Uniqueness and integral formula: Let $P \in \mathbb{C}^{n \times n}$ be a projection that satisfies (a) and (b) and let the complex path γ have the properties specified at the end of the proposition. It suffices to show that P is given by the claimed path integral.

Since P commutes with A, it also commutes with $(\mu - A)^{-1}$ for every $\mu \in \rho(A)$ and one has $(\mu - A)^{-1}|_{\operatorname{rg} P} = (\mu - A|_{\operatorname{rg} P})^{-1}$ for all such μ; the same also holds for the restriction to $\ker P$. So Corollary 2.1.3(a) applied to $A|_{\operatorname{rg} P}$

$$\frac{1}{2\pi \mathrm{i}} \oint_\gamma (\mu - A)^{-1} \,\mathrm{d}\mu|_{\operatorname{rg} P} = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (\mu - A|_{\operatorname{rg} P})^{-1} \,\mathrm{d}\mu = \mathrm{id}_{\operatorname{rg} P} = P|_{\operatorname{rg} P}$$

and similarly, Corollary 2.1.3(b) applied to $A|_{\ker P}$ gives

$$\frac{1}{2\pi \mathrm{i}} \oint_\gamma (\mu - A)^{-1} \,\mathrm{d}\mu|_{\ker P} = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (\mu - A|_{\ker P})^{-1} \,\mathrm{d}\mu = 0 = P|_{\ker P}.$$

This shows the claimed formula for P. ☐

Definition 2.1.5 (Spectral projections)**.** In the situation of Proposition 2.1.4, the projection P is called the **spectral projection** of A associated to σ_0.

In the situation of Proposition 2.1.4, note that the complementary projection $1 - P$ also commutes with A and satisfies $\operatorname{rg}(1 - P) = \ker P$ and $\ker(1 - P) = \operatorname{rg} P$. Hence, it follows that $1 - P$ is the spectral projection of A associated to $\sigma(A) \setminus \sigma_0$.

Recall that an eigenvalue λ of a square matrix A is called **semisimple** if its geometric and algebraic multiplicities coincide.

Proposition 2.1.6. *Let $A \in \mathbb{C}^{n \times n}$ and let P be the spectral projection of A associated to an eigenvalue $\lambda \in \sigma(A)$.*[1]

(a) *P is equal to the coefficient Q_0 of the term $(\mu - \lambda)^{-1}$ in the Laurent series expansion of $(\mu - A)^{-1}$ in Proposition 2.1.1(a).*

(b) $\operatorname{rg} P = \bigcup_{k=1}^n \ker(\lambda - A)^k$, *i.e. the range of P coincides with the generalised eigenspace of λ. In particular,* $\dim \operatorname{rg} P$ *is the algebraic multiplicity of the eigenvalue λ.*

(c) *The following are equivalent:*

  (i) *The eigenvalue λ is a first order pole of the resolvent $\mathcal{R}(\cdot, A)$.*

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[1] This is an informal way of saying that P is the spectral projection associated to the set $\{\lambda\}$.

(ii) *The eigenvalue λ is a first order pole of the dual resolvent $\mathcal{R}(\cdot, A^{\mathrm{T}})$.*

(iii) *The limit $\lim_{\mu\to\lambda}(\mu-\lambda)\mathcal{R}(\mu, A)$ exists.*

(iv) *The eigenvalue λ is semisimple.*

(v) *The range of P consists of eigenvectors, i.e. $\operatorname{rg} P = \ker(\lambda - A)$.*

If the equivalent conditions (i)–(v) are satisfied, then $\lim_{\mu\to\lambda}(\mu-\lambda)\mathcal{R}(\mu, A) = P$.

(d) *If $\ker(\lambda - A) = \operatorname{span}\{u\}$ and $v \in \ker(\lambda - A^{\mathrm{T}})$ satisfy $v^{\mathrm{T}}u = 1$, then $P = uv^{\mathrm{T}}$.*

Proof. (a) This follows from the integral representation of P in Proposition 2.1.4 and from the integration formula in Proposition 2.1.2(a).

(b) This follows from how P was constructed in the existence argument in the proof of Proposition 2.1.4.

(c) If λ is a first order pole, then $\lim_{\mu\to\lambda}(\mu-\lambda)\mathcal{R}(\mu, A) = Q_0$ by the Laurent series expansion of $\mathcal{R}(\cdot, A)$ about λ (Proposition 2.1.1(a)). Moreover, $Q_0 = P$ according to (a). Let us now prove the claimed equivalence.

"(i) ⇔ (ii)": One has $\mathcal{R}(\mu, A^{\mathrm{T}}) = \mathcal{R}(\mu, A)^{\mathrm{T}}$ for all $\mu \in \rho(A^{\mathrm{T}}) = \rho(A)$, so one can see the claimed equivalence by taking the transposition operation out of the Laurent series expansion in Proposition 2.1.1(a).

"(i) ⇔ (iii)": For a scalar-valued holomorphic function f that has an isolated singularity at λ it is a standard result from complex analysis that λ is a first order pole of f if and only if $\lim_{\mu\to\lambda}(\mu-\lambda)f(\mu)$ exists. The claim now follows from applying this fact to all components of the resolvent.

"(i) ⇔ (iv)": Since λ is semisimple if and only if every Jordan block corresponding to λ has size 1, which is equivalent to λ having pole order one by Proposition 2.1.1.

"(i) ⇒ (v)": Note that $\ker(\lambda - A) \subseteq \operatorname{rg} P = \operatorname{rg} Q_0$ by (b) and (a). If the pole order at λ is 1, the converse inclusion $\operatorname{rg} Q_0 \subseteq \ker(\lambda - A)$ follows from Proposition 2.1.1(b).

"(v) ⇒ (iv)": The geometric multiplicity of λ is $\dim\ker(\lambda - A)$ and the algebraic multiplicity of λ is $\dim \operatorname{rg} P$, according to (b). It follows from (v) that the two are equal.

(d) The assumptions ensure that λ is even algebraically simple (hence, semisimple) due to Lemma 1.2.8. Now (c) implies $\operatorname{rg} P$ is spanned by u. By applying the same argument to A^{T} – whose spectral projection for the eigenvalue λ is P^{T} due to formula (2.1.4) – we also see that $\operatorname{rg} P^{\mathrm{T}}$ is spanned by v. Thus, $P = \alpha uv^{\mathrm{T}}$ for a scalar α. As P is a projection and $v^{\mathrm{T}}u = 1$, it follows that $\alpha = 1$. ☐

Theorem 2.1.7 (Spectral mapping theorem for the matrix exponential function)**.** *Let $A \in \mathbb{C}^{n\times n}$ and let $t \in \mathbb{R}$. Then one has $\sigma(e^{tA}) = e^{t\sigma(A)}$. More precisely:*

(a) *If $\lambda \in \mathbb{C}$ is an eigenvalue of A with eigenvector $z \in \mathbb{C}^n$, then $e^{t\lambda}$ is an eigenvalue of e^{tA} with eigenvector z.*

(b) *If $\mu \in \sigma(e^{tA})$, then there exists $\lambda \in \sigma(A)$ such that $\mu = e^{t\lambda}$.*

Proof. (a) It follows from $Az = \lambda z$ that $(tA)^j z = (t\lambda)^j z$ for all integer $j \geq 0$, hence

$$e^{tA} z = \sum_{j=0}^{\infty} \frac{(tA)^j}{j!} z = \sum_{j=0}^{\infty} \frac{(t\lambda)^j}{j!} z = e^{t\lambda} z.$$

(b) After a coordinate transformation, we may assume A is in Jordan normal form (2.1.1). Then by Exercise 1.4(b), e^{tA} is an upper triangular matrix whose eigenvalues are $e^{t\lambda_1}, \dots, e^{t\lambda_m}$, where $\lambda_1, \dots, \lambda_m$ are the eigenvalues of A (counted with their geometric multiplicity). Hence $\mu = e^{t\lambda}$ for some $\lambda \in \sigma(A)$. ☐

2.2 Eventually positive matrix semigroups

Our main objects of study for the rest of Chapter 2 are matrix semigroups $(e^{tA})_{t\geq 0}$ which are positive for all sufficiently large times.

Definition 2.2.1 (Eventually (strongly) positive matrix semigroups)**.** Let $A \in \mathbb{R}^{n\times n}$.

(a) The matrix semigroup $(e^{tA})_{t\geq 0}$ is called **eventually positive** if there exists $t_0 \geq 0$ such that $e^{tA} \geq 0$ for all $t \in [t_0, \infty)$.

(b) The matrix semigroup $(e^{tA})_{t\geq 0}$ is called **eventually strongly positive** if there exists $t_0 \geq 0$ such that $e^{tA} x \succeq \mathbb{1}$ for all $0 \neq x \in \mathbb{R}^n_+$ and all $t \in [t_0, \infty)$.

This definition uses Notation 1.2.4 again, i.e. for a given t, the inequality $e^{tA} x \succeq \mathbb{1}$ means that there exists a number $c > 0$ such that $e^{tA} x \geq c\,\mathbb{1}$. Observe that c can a priori depend on t.

Examples 2.2.2.

(a) The matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is nilpotent and satisfies $A^k = 0$ for all $k \geq 3$. Moreover,

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{so} \qquad e^{tA} = \begin{pmatrix} 1 & t & \frac{t^2}{2} - t \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

for all $t \geq 0$. Thus, $(e^{tA})_{t\geq 0}$ is eventually positive but not eventually strongly positive. Since $\frac{t^2}{2} - t < 0$ for $t \in (0, 2)$, the semigroup is not positive. Alternatively, this follows from the characterisation of positive semigroups in Theorem 1.3.8 since the off-diagonal entry $A_{13} = -1$ is strictly negative.

(b) Consider the matrix

$$A := U \begin{pmatrix} 0 & & \\ & -1 & \\ & & -9 \end{pmatrix} U^{\mathrm{T}} = \begin{pmatrix} -2 & -1 & 3 \\ -1 & -2 & 3 \\ 3 & 3 & -6 \end{pmatrix},$$

where $U := (u_1, u_2, u_3) \in \mathbb{R}^{3\times 3}$ is the orthogonal matrix with the columns

$$u_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}, \quad u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^{\mathrm{T}}, \quad u_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & -2 \end{pmatrix}^{\mathrm{T}}.$$

Let $P \in \mathbb{R}^{3\times 3}$ be the projection onto the first component of $\mathbb{R}^3$. Then $e^{tA} \to UPU^{\mathrm{T}} = u_1 u_1^{\mathrm{T}}$ as $t \to \infty$. Since every entry of $u_1 u_1^{\mathrm{T}}$ is $\frac{1}{3}$, it follows that $(e^{tA})_{t\geq 0}$ is eventually strongly positive. However as A has some negative off-diagonal entries, so by the characterisation of positive semigroups in Theorem 1.3.8, $(e^{tA})_{t\geq 0}$ is not positive.

Parts (a) and (b) of the following result show that Theorem 1.3.9 about positive semigroups continues to hold in the eventually positive case. We also add a third property (c) which we will use to study perturbation theory in Theorem 2.4.2.

Theorem 2.2.3 (Perron–Frobenius for eventually positive matrix semigroups)**.** *Let $A \in \mathbb{R}^{n\times n}$ be such that $(e^{tA})_{t\geq 0}$ is eventually positive. The following assertions hold:*

(a) *The spectral bound $\mathrm{s}(A)$ is an eigenvalue of A with an eigenvector $x \geq 0$.*

(b) *$\mathrm{s}(A)$ is a strictly dominant eigenvalue of A, i.e. $\operatorname{Re}\lambda < \mathrm{s}(A)$ for all $\lambda \in \sigma(A) \setminus \{\mathrm{s}(A)\}$.*

(c) *If $\mathrm{s}(A)$ is semisimple, then its associated spectral projection is positive.*

Proof. By replacing A with $A - \mathrm{s}(A)$ (which does not affect questions of positivity), we may assume that $\mathrm{s}(A) = 0$. Let $t_0 \in [0,\infty)$ be such that $e^{tA} \geq 0$ for all $t \geq t_0$.

(a) and (b) We first prove that 0 is an eigenvalue of A and that there are no non-zero eigenvalues of A on the imaginary line. Since $\mathrm{s}(A) = 0$, it follows from the spectral mapping theorem for the matrix exponential function (Theorem 2.1.7) that e^{tA} has spectral radius 1 for each $t \geq 0$. Let $i\beta \in i\mathbb{R}$ be an eigenvalue of A. Again by Theorem 2.1.7, $e^{i\beta t}$ is an eigenvalue of e^{tA} for each $t \in [0,\infty)$. We need to show that $\beta = 0$.

For each index $j \in \{1,\dots,n\}$ one has $(e^{0A})_{jj} = 1$ hence by the uniqueness theorem for analytic functions, the set $\bigcup_j \left\{t \in [0,\infty) : (e^{tA})_{jj} = 0\right\}$ does not accumulate in $[0,\infty)$. Thus, there exist times $t_2 > t_1 \geq t_0$ such that for each $t \in [t_1, t_2]$, the diagonal entries of e^{tA} are strictly positive. According to Theorem 1.2.5(b) this implies, for each such t, that the spectral radius 1 is a radially strictly dominant eigenvalue of e^{tA}, i.e. the matrix e^{tA} does not have eigenvalues on the unit circle except for the number 1. Thus, $e^{i\beta t} = 1$ for all $t \in [t_1, t_2]$, which implies that $\beta = 0$, as claimed.

Now we show the existence of a positive eigenvector of A for the eigenvalue 0. With the notation of the Laurent series expansion of $(\mu - A)^{-1}$ about the eigenvalue 0 in

Proposition 2.1.1(a), one gets $Q_{-p+1} = \lim_{\mu\to 0} \mu^p(\mu - A)^{-1}$. Using the Laplace transform representation of the resolvent (Lemma 1.3.7) yields

$$Q_{-p+1} = \lim_{\mu\downarrow 0}\Big(\underbrace{\mu^p\int_0^{t_0} e^{-t\mu}e^{tA}\,\mathrm{d}t}_{\to 0} + \underbrace{\mu^p\int_{t_0}^{\infty} e^{-t\mu}e^{tA}\,\mathrm{d}t}_{\geq 0}\Big) \geq 0.$$

Since Q_{-p+1} is non-zero and $\mathbb{R}^n_+$ spans $\mathbb{R}^n$, we can find a vector $x \in \mathbb{R}^n_+$ such that $0 \leq Q_{-p+1}x \neq 0$. According to Proposition 2.1.1(b) that range of Q_{-p+1} is contained in $\ker A$, so $Q_{-p+1}x$ is indeed a positive eigenvector of A for the eigenvalue $\mathrm{s}(A) = 0$.

(c) Due to semisimplicity, Proposition 2.1.6(a) and (c) give $P = Q_0 = Q_{-p+1} \geq 0$. □

Example 2.2.4. In Theorem 2.2.3(c), the spectral projection can fail to be positive if $\mathrm{s}(A)$ is not a semisimple eigenvalue. Indeed, let

$$A := T\begin{pmatrix}0&1&0\\0&0&0\\0&0&-2\end{pmatrix}T^{-1} = \begin{pmatrix}-1&1&0\\1&-1&0\\1&0&0\end{pmatrix} \quad\text{for}\quad T := \begin{pmatrix}0&1&2\\0&1&-2\\1&0&-1\end{pmatrix}$$

Since all off-diagonal entries of A are ≥ 0, the semigroup $(e^{tA})_{t\geq 0}$ is positive (Theorem 1.3.8). The given Jordan normal form of A shows that $\mathrm{s}(A) = 0$ is not semisimple. The spectral projection of A associated to the eigenvalue 0 is

$$P = T\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}T^{-1} = \frac{1}{4}\begin{pmatrix}2&2&0\\2&2&0\\1&-1&4\end{pmatrix} \ngeq 0.$$

2.3 Characterisation

By Theorem 1.3.8, positivity of $(e^{tA})_{t\geq 0}$ is equivalent to the positivity of $\mathcal{R}(\lambda, A)$ for all $\lambda \in (\mathrm{s}(A), \infty)$. Parts (i) and (ii) of the next theorem provide a related characterisation for eventual strong positivity. Parts (iii) and (iv) show that eventual strong positivity can be characterised in terms of Perron–Frobenius like spectral properties. In this sense, Perron–Frobenius theory is closer to eventual positivity than to positivity.

Theorem 2.3.1. *Let $A \in \mathbb{R}^{n\times n}$. The following are equivalent:*

(i) $(e^{tA})_{t\geq 0}$ *is eventually strongly positive.*

(ii) $\mathrm{s}(A)$ *is a strictly dominant eigenvalue of A and there exists $\lambda_0 > \mathrm{s}(A)$ such that $\mathcal{R}(\lambda, A)$ is strongly positive for all $\lambda \in (\mathrm{s}(A), \lambda_0)$.*

(iii) $\mathrm{s}(A)$ *is a strictly dominant eigenvalue of A and the associated spectral projection is strongly positive.*

(iv) $\mathrm{s}(A)$ *is a strictly dominant eigenvalue of A and the eigenspaces* $\ker(\mathrm{s}(A) - A)$ *and* $\ker(\mathrm{s}(A) - A^{\mathrm{T}})$ *are spanned by strongly positive vectors.*

If these equivalent assertions hold, then the eigenvalue $\mathrm{s}(A)$ *is even algebraically simple.*

Proof. As usual, we may assume $\mathrm{s}(A) = 0$. Note that if the spectral projection P associated to 0 is strongly positive, then the eigenvalue 0 of A is algebraically simple (and in turn, semisimple): indeed, if P is strongly positive, then Perron–Frobenius (Theorem 1.2.5(c)) guarantees that $\mathrm{r}(P) > 0$ and $\ker(\mathrm{r}(P) - P)$ is spanned by a strongly positive vector. On the other hand, $\mathrm{r}(P) = 1$, as P is a non-zero projection. Consequently, $\operatorname{rg} P = \ker(1 - P)$ is one-dimensional. As $\dim \operatorname{rg} P$ is the algebraic multiplicity of the eigenvalue $\mathrm{s}(A) = 0$ (Proposition 2.1.6(b)), it follows that 0 is algebraically simple.

"(i) ⇒ (iv)": By Theorem 2.2.3 (Perron–Frobenius for eventually positive matrix semigroups) the spectral bound 0 is a strictly dominant eigenvalue of A. Choose $t_0 > 0$ such that the matrix $e^{t_0 A}$ is strongly positive. Due to the spectral mapping theorem for the matrix exponential function (Theorem 2.1.7(a)), one has $\{0\} \neq \ker A \subseteq \ker(1 - e^{t_0 A})$, and the latter space is spanned by a strongly positive vector according to the Perron–Frobenius theorem for strongly positive matrices (Theorem 1.2.5(c)). Thus, $\ker A = \ker(1 - e^{t_0 A})$, which proves the claim for $\ker A$. The same argument applies to $\ker(A^{\mathrm{T}})$, since $e^{t_0 A^{\mathrm{T}}} = (e^{t_0 A})^{\mathrm{T}}$ is also strongly positive.

"(iv) ⇒ (iii)": By assumption, $\ker A$ and $\ker A^T$ are spanned by strongly positive vectors u, v respectively. Replacing u by a scalar multiple, we may assume that $v^{\mathrm{T}} u = 1$. Proposition 2.1.6(d) now yields $P = uv^{\mathrm{T}}$ is strongly positive.

"(iii) ⇒ (i)": We have seen that the strong positivity of P implies that 0 is semisimple. This ensures $\operatorname{rg} P = \ker A$ due to Proposition 2.1.6(c). The spectral mapping theorem (Theorem 2.1.7) thus implies that e^{tA} acts as the identity matrix on $\operatorname{rg} P$.

Also, since 0 is a strictly dominant eigenvalue of A, all eigenvalues of $A|_{\ker P}$ have strictly negative real part by Proposition 2.1.4(b). Therefore, $e^{tA}|_{\ker P} \to 0$ as $t \to \infty$ according to Proposition 1.3.6. Consequently, $e^{tA} = e^{tA}P + e^{tA}(1 - P) \to P$ as $t \to \infty$. The strong positivity of P hence implies the eventual strong positivity of $(e^{tA})_{t \geq 0}$.

"(ii) ⇒ (iv)": Let $\lambda \in (0, \lambda_0) = (\mathrm{s}(A), \lambda_0)$. One has

$$\sigma\big(\mathcal{R}(\lambda, A)\big) = \left\{ \frac{1}{\lambda - \mu} : \mu \in \sigma(A) \right\}.$$

As $\mathrm{s}(A) = 0 \in \sigma(A)$, it follows that $\mathrm{r}(\mathcal{R}(\lambda, A)) = \frac{1}{\lambda}$ is an eigenvalue of $\mathcal{R}(\lambda, A)$.

Observe that $\ker A = \ker\big(\frac{1}{\lambda} - \mathcal{R}(\lambda, A)\big)$ and the latter space is spanned by a strongly positive vector according to the Perron–Frobenius theorem for strongly positive matrices (Theorem 1.2.5(c)). The same argument can be applied to A^{T}.

"(iii) ⇒ (ii)": Since P is strongly positive, 0 is semisimple as already shown. Proposition 2.1.6(c) thus gives $\lim_{\mu \to 0} \mu \mathcal{R}(\mu, A) = P$. As P is strongly positive, this implies that $\mathcal{R}(\mu, A)$ is also strongly positive for all $\mu > 0$ that are sufficiently close to 0. □

2.4 Perturbations

We conclude this chapter with a sneak peek of the perturbation theory for eventually positive semigroups. By **perturbations** – more precisely, additive perturbations – we mean the following: given two matrices $A, B \in \mathbb{C}^{n\times n}$, we study which properties of the semigroup $(e^{tA})_{t\geq 0}$ are inherited by the **perturbed** semigroup $\left(e^{t(A+B)}\right)_{t\geq 0}$ if B has sufficiently nice properties. In other words, B is viewed as a perturbation of A , and our goal is to determine which semigroup properties are preserved under such perturbations.

A simple instance of such a perturbation result is the fact that positive perturbations do not destroy the positivity of a semigroup. This is a particular case of the following.

Proposition 2.4.1. *Let $A, B \in \mathbb{R}^{n\times n}$ and assume that $(e^{tA})_{t\geq 0}$ is positive. If all off-diagonal entries of B are ≥ 0, then the perturbed semigroup $\left(e^{t(A+B)}\right)_{t\geq 0}$ is also positive.*

Proof. By Theorem 1.3.8, the semigroup generated by a matrix $C \in \mathbb{R}^{n\times n}$ is positive if and only if all off-diagonal entries of C are ≥ 0. The assertion is now immediate. ☐

It is natural to ask whether a similar perturbation result holds for eventually positive semigroups: if $(e^{tA})_{t\geq 0}$ is eventually positive and $B \in \mathbb{R}^{n\times n}$ is a positive matrix, does it follow that the perturbed semigroup $\left(e^{t(A+B)}\right)_{t\geq 0}$ is also eventually positive? The answer to this question is quite surprising (and perhaps disappointing): unless the unperturbed semigroup is already positive, there always exists a positive perturbation that destroys the eventual positivity. We prove this in the following theorem.

Theorem 2.4.2. *Let $A \in \mathbb{R}^{n\times n}$. The following are equivalent.*

(i) *For every $B \in \mathbb{R}_+^{n\times n}$, the semigroup $\left(e^{t(A+B)}\right)_{t\geq 0}$ is eventually positive.*

(ii) *For every $B \in \mathbb{R}_+^{n\times n}$ of rank ≤ 1, the semigroup $\left(e^{t(A+B)}\right)_{t\geq 0}$ is eventually positive.*

(iii) *The semigroup $(e^{tA})_{t\geq 0}$ is positive.*

Proof. *"(iii) ⇒ (i)":* In this case, $\left(e^{t(A+B)}\right)_{t\geq 0}$ is even positive (Proposition 2.4.1).

"(i) ⇒ (ii)": This is trivial.

"(ii) ⇒ (iii)": This is the surprising part. The key ingredient is the Sherman–Morrison–Woodbury formula for rank-one perturbations of resolvents, presented in Exercise 2.3.

As before, we assume $\operatorname{s}(A) = 0$ without loss of generality. According to Theorem 1.3.8, it suffices to prove that $\mathcal{R}(\mu, A) \geq 0$ for all $\mu > 0$. To achieve this, in fact it suffices to prove that $v^{\mathrm{T}}\mathcal{R}(\mu, A) \geq 0$ for all $v \geq \mathbb{1}$ and all $\mu > 0$, since strongly positive vectors are dense in $\mathbb{R}_+^n$. Thus let us fix such a vector $v \geq \mathbb{1}$ and a number $\mu > 0$.

Firstly, observe that assumption (ii) with $B = 0$ implies that $(e^{tA})_{t\geq 0}$ is eventually positive. By Theorem 2.2.3, we deduce that $\operatorname{s}(A) = 0$ is an eigenvalue of A with an eigenvector $u \geq 0$. Thus $v^{\mathrm{T}} u > 0$ and there exists $\alpha > 0$ such that $\alpha v^{\mathrm{T}} u = \mu$.

Consider the rank-one matrix $B := \alpha u v^{\mathrm{T}}$. By Exercise 2.3(b), we have $\mathrm{s}(A+\alpha u v^{\mathrm{T}}) = \mu$, it is a semisimple eigenvalue of $A+B$, and formula (2.4.2) (with $\lambda_0 = 0$) yields

$$(\lambda-\mu)\mathcal{R}(\lambda, A+\alpha u v^{\mathrm{T}}) = (\lambda-\mu)\mathcal{R}(\lambda, A) + \alpha u v^{\mathrm{T}}\mathcal{R}(\lambda, A)$$

for all $\lambda > \mu$. Due to semisimplicity, Proposition 2.1.6(c) ensures that the spectral projection corresponding to the eigenvalue μ of $A+B$ is given by

$$\lim_{\lambda\downarrow\mu}(\lambda-\mu)\mathcal{R}(\lambda, A+\alpha u v^{\mathrm{T}}) = \alpha u v^{\mathrm{T}}\mathcal{R}(\mu, A).$$

By hypothesis, $A+B$ generates an eventually positive semigroup, so this projection is positive by Theorem 2.2.3(c). As $u \geq 0$ and non-zero, it follows that $v^{\mathrm{T}}\mathcal{R}(\mu, A) \geq 0$. ☐

Theorem 2.4.2 is not quite the end of the story. The notion “perturbation” already suggests that one is often interested in perturbations that are small in some sense. Furthermore, the above theorem leaves open the possibility that perhaps a more positive result (pun intended) holds for eventual *strong* positivity. As it turns out, one can show that eventual strong positivity of a semigroup $(e^{tA})_{t\geq 0}$ is preserved by all perturbations $B \geq 0$ that are sufficiently small in operator norm. In fact, such a result holds even in infinite dimensions, as we will see in Chapter 13, where perturbation theory for eventually positive semigroups is developed in greater depth.

Exercises for Chapter 2

Exercise 2.1. Consider the matrix

$$A := T \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} T^{-1} \in \mathbb{R}^{4\times 4}$$

for an invertible matrix $T \in \mathbb{R}^{4\times 4}$.

(a) Compute the Laurent series expansion of $\mathcal{R}(\cdot, A)$ about the spectral value 1 and the associated spectral projection. What is the pole order of $\mathcal{R}(\cdot, A)$ at 1?

(b) Compute the Laurent series expansion of $\mathcal{R}(\cdot, A)$ about the spectral value 0 and the associated spectral projection. What is the pole order of $\mathcal{R}(\cdot, A)$ at 0?

Does one have equality of the subspaces in Proposition 2.1.1(b)?

(c) Find a T such that $(e^{tA})_{t\geq 0}$ is eventually strongly positive.

Exercise 2.2. Prove the following assertions are equivalent for $A \in \mathbb{R}^{n\times n}$:

(i) $(e^{tA})_{t\geq 0}$ is eventually positive.

(ii) For every $0 \leq x \in \mathbb{R}^n$, there exists $t_0 = t_0(x) \geq 0$ such that $e^{tA}x \geq 0$ for all $t \geq t_0$.

(iii) For every $0 \leq x \in \mathbb{R}^n$ and $0 \leq y \in \mathbb{R}^n$, there exists $t_0 = t_0(x, y) \geq 0$ such that $y^{\mathrm{T}} e^{tA} x \geq 0$ for all $t \geq t_0$.

Exercise 2.3 (Sherman–Morrison–Woodbury formula)**.** Let $A \in \mathbb{C}^{n\times n}$ and let $u, v \in \mathbb{C}^n$.

(a) If A is invertible, prove that $A - uv^{\mathrm{T}}$ is invertible if and only if $v^{\mathrm{T}} A^{-1} u \neq 1$, and in this case it holds that

$$(A - uv^{\mathrm{T}})^{-1} = A^{-1} + \frac{1}{1 - v^{\mathrm{T}} A^{-1} u} A^{-1} u v^{\mathrm{T}} A^{-1}. \tag{2.4.1}$$

(b) If $\lambda \in \rho(A)$, and u is an eigenvector corresponding to an eigenvalue $\lambda_0 \in \mathbb{C}$ of A, deduce that $\lambda - (A + uv^{\mathrm{T}})$ is invertible if and only if $v^{\mathrm{T}} u \neq \lambda - \lambda_0$, and in this case

$$\mathcal{R}(\lambda, A + uv^{\mathrm{T}}) = \mathcal{R}(\lambda, A) + \frac{1}{(\lambda - \lambda_0) - v^{\mathrm{T}} u} u v^{\mathrm{T}} \mathcal{R}(\lambda, A). \tag{2.4.2}$$

Moreover, deduce that $\lambda_0 + v^{\mathrm{T}} u$ is a semisimple eigenvalue of $A + uv^{\mathrm{T}}$.

Exercise 2.4 (Another characterisation of eventual strong positivity)**.** Let $A \in \mathbb{R}^{n \times n}$.

(a) Assume that there exists $c \in \mathbb{R}$ and $k_0 \in \mathbb{N}$ such that $(A + cI)^k$ is strongly positive for all integers $k \geq k_0$. Show that $(e^{tA})_{t \geq 0}$ is eventually strongly positive.

(b) Suppose $B \in \mathbb{C}^{n \times n}$ is a matrix such that $\mathrm{r}(B) > 0$ is a semisimple and radially strictly dominant eigenvalue (see Theorem 1.2.5(b)). Prove that $\left(\frac{B}{\mathrm{r}(B)}\right)^k$ converges to the spectral projection associated with $\mathrm{r}(B)$ as $k \to \infty$. [*Hint:* Jordan normal form.]

(c) Assume that $(e^{tA})_{t \geq 0}$ is eventually strongly positive. Use part (b) to deduce that there exists $k_0 \in \mathbb{N}$ and $c \in \mathbb{R}$ such that $(A + cI)^k$ is strongly positive for all $k \geq k_0$.

Exercise 2.5 (Eventual (?) positivity in two dimensions)**.** Let $A \in \mathbb{R}^{2 \times 2}$. Show that if $(e^{tA})_{t \geq 0}$ is eventually (strongly) positive, then $(e^{tA})_{t \geq 0}$ is (strongly) positive.

Hint: as a first step, think about what $\sigma(A)$ could look like.

Notes for Chapter 2

Spectral projections

For the historical development of the functional calculus for linear operators – which contains spectral projections as a special case – we refer to Section 5.2.1 in Pietsch's monograph [Pie07] about the history of Banach spaces and linear operators. A very accessible presentation of spectral projections of matrices and, more generally, of eigenvalue theory via complex analysis techniques can be found, for instance, in [CD13] (an updated version with minor corrections is available on Daniel Daners' webpage).

Eventual positivity in finite dimensions

Matrices with eventually positive powers

The predecessors of the eventual positivity theory in finite dimensions can be found in various results about matrices A whose powers A^k are positive for some, or all sufficiently large, $k \in \mathbb{N}$. Matrices with a positive power were, for instance, studied in [Bra61], and more recently in [TCDF15]. Matrices for which a polynomial $p(A)$ is positive are studied in [Sen06]. Some early papers on matrices with eventually positive powers, such as [Fri78, ZT99], were motivated by inverse spectral problems, i.e. the question of which sets in $\mathbb{C}$ can be realised as the spectrum of matrices with certain prescribed properties.

In the 21st century, the literature on eventually positive matrices has grown quickly. Of particular interest were spectral properties of such matrices in the spirit of the Perron–Frobenius theorem, e.g. in the papers [TRH01, JT04, Nou06, ES08, ES09]. In particular, in [Nou06, Theorem 2.2] one can find a discrete-time analogue of the equivalence of (i) and (iv) in Theorem 2.3.1. Matrices with eventually positive powers did not occur in the lecture notes, but they appear in Exercise 2.4. For further references about matrices with eventually positive powers we refer to [Glü16, Section 6.4]; most of the preceding two paragraphs is also taken from this reference.

Eventually positive matrix semigroups

As for the continuous-time case, eventual positivity of matrix semigroups was studied by Noutsos and Tsatsomeros in [NT08]. The equivalence of (i) and (iv) in Theorem 2.3.1 as well as the characterisation of eventually strongly positive matrix semigroups in Exercise 2.4 appeared in [NT08, Theorem 3.3]; however, the approach outlined in the exercise follows [DGK16, Theorem 6.1]. The fact that eventual positivity implies positivity for 2×2 matrix semigroups (Exercise 2.5) was observed in [DGK16, Proposition 6.2].

Perturbation theory

Perturbation theory for matrices with eventually positive powers was studied in [SA17]. In particular, [SA17, Proposition 3.6 and Theorem 3.7] contain discrete-time analogues of the equivalence of (i) and (iii) in Theorem 2.4.2. For the continuous-time case, perturbation theory was first studied in [DG18], though with a focus on the infinite-dimensional case that we consider later. A specific finite-dimensional result in this article is [DG18, Proposition 4.6], which shows that the set of all $A \in \mathbb{R}^{n\times n}$ for which $(e^{tA})_{t\geq 0}$ is eventually strongly positive is open in $\mathbb{R}^{n\times n}$. Theorem 2.4.2 is a finite-dimensional version of [DG18, Theorem 2.3]; we were able to slightly weaken the assumptions in this theorem. Example 2.2.2(b) is also taken from [DG18, Example 2.1]. There, an explicit rank-one operator B is given with the property that $A+sB$ generates an eventually positive semigroup for $s \in [0,4)$, but the eventual positivity is lost for $s > 4$.

Eventual positivity with respect to other cones

Positivity of matrices and matrix semigroups has often been studied with respect to general cones. Naturally, this can also be done for eventual positivity; see, for instance, [Kas17, KT17, Soo19]. The following phenomenon in this context is remarkable: Exercise 2.2 suggests that eventual positivity for a dynamical system can be defined 'individually' (i.e. for individual orbits $x \mapsto e^{tA}x$) or 'uniformly' on the level of operators (as in Definition 2.2.1). However, the result of that exercise states that these two notions are equivalent for matrix semigroups. This is a product of two separate features: finite dimensionality, and geometric properties of the positive cone $\mathbb{R}^n_+$. As we see later, the equivalence of individual and uniform eventual positivity fails in infinite dimensions. More surprisingly, it also fails in finite dimensions if we consider positivity with respect to other cones, as was shown in [GH23, Example 3.1].

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